Lectures on

FOURIER ANALYSIS

BY

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1. Fourier series on the circle group

Let S^1 stand for the set of all complex numbers z of absolute value one. This becomes a group under multiplication which is called the circle group. Any element z of this group can be written as $z = e^{2\pi it}$ for a unique $t \in [0,1)$. In view of this we can identify S^1 with $[0,1)$ and there is a one to one correspondence between functions on S^1 and functions on the real line $\mathbb{R}$ that are $1-$periodic, i.e. functions f satisfying $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. The most distinguished functions on S^1 are the trigonometric functions e_k defined by

$$e_k(t) = (\cos 2\pi kt + i \sin 2\pi kt) = e^{2\pi ikt}.$$

Here $k \in \mathbb{Z}$ the set of all integers. These functions are distinguished for several reasons.

First of all they are elementary, smooth functions which have been studied from ancient times. They are eigenfunctions of the one dimensional Laplacian:

$$\frac{d^2}{dt^2} e_k(t) = -4\pi^2 k^2 e_k(t).$$

For each $k \in \mathbb{Z}$ the map

$$\chi_k : S^1 \to S^1, \ \chi_k(e^{2\pi it}) = e_k(t)$$

is a homomorphism. Moreover, for each $t \in \mathbb{R}$ the map

$$\varphi_t : \mathbb{Z} \to S^1, \ \varphi_t(k) = e_k(t)$$

is also a homomorphism. They form an orthonormal system in the sense that

$$(e_k, e_j) = \int_0^1 e_k(t)\bar{e_j}(t)dt = \delta_{k,j}.$$

Above all, they are the building blocks using which we can construct all functions on the circle group.

1.1. **Fourier series of continuous functions.** We let $C(S^1)$ to stand for the Banach space of all continuous functions on S^1 equipped with the norm $\|f\|_\infty = sup_{0\leq t\leq 1}|f(t)|$. Note that we have identified the function f on S^1 with a $1-$periodic function on $\mathbb{R}$. We do this identification without any further comments. Given $f \in C(S^1)$ we have the formal power series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e_k.$$

We are interested in knowing if the partial sums S_nf defined by

$$S_nf(t) = \sum_{k=-n}^{n} \hat{f}(k)e_k(t)$$

converge to the function f as n tends to infinity. The choice of the symmetric partial sums instead of more general partial sums is dictated by the fact that both e_k and e_{-k} are eigenfunctions of $\frac{d^2}{dt^2}$ with the same eigenvalue.

In du Bois Reymond constructed a continuos function whose Fourier series diverges at a given point. This can also be proved by appealing to uniform boundedness principle. However, we have a positive result which asserts that trigonometric polynomials are dense in $C(S^1)$. By the term trigonometric polynomials we mean functions of the form $p_n(t) = \sum_{k=-n}^{n} a_n(k)e_k(t)$. Let $\mathcal{P}(S^1)$ stand for the space of all such polynomilas. The space $C(S^1)$ is also an algebra under pointwise multiplication of functions and $\mathcal{P}(S^1)$ is clearly a subalgebra which satisfies the following conditions: (i) it separates points on S^1; (ii) it contains all the constants; (iii) it is self-adjoint in the sense that it is closed under conjugation. By appealing to Stone-Weierstrass theorem for the compact Hausdorff space S^1 we get

Theorem 1.1. *Trigonometric polynomials are dense in $C(S^1)$.*

This theorem does not give any clue for explicitly constructing a sequence of trigonometric polynomials that converges to the given function f. Suppose the sequence $p_n(t) = \sum_{k=-n}^{n} a_n(k)e_k(t)$ converges to f in $C(S^1)$. Then the kth Fourier coefficient of p_n, namely $a_n(k)$ converges to $\hat{f}(k)$. This is the only information we can get on p_n. In the Hungarian mathematician Fejer proved a very elegant result by explicitly constructing a sequence $\sigma_n f$ of trigonometric polynomials which converges to f in $C(S^1)$. His idea is just to take the arithmetic means of the partial sums. Thus he defined

$$\sigma_n f = \frac{1}{n+1}\sum_{k=0}^{n} S_k f.$$

It is clear that $\sigma_n f$ are trigonometric polynomials. He was then able to show that $\sigma_n f$ converges to f in $C(S^1)$.

We define the convolution of two functions f and g on S^1 by

$$f * g(t) = \int_0^1 f(t-s)g(s)ds = \int_0^1 f(s)g(t-s)ds.$$

The second equality is a consequence of the fact that for a $1-$periodic function h,

$$\int_0^1 h(s)ds = \int_a^{a+1} h(s)ds$$

for any $a \in \mathbb{R}$. We observe that

$$f * e_k(t) = \int_0^1 f(s)e_k(t-s)ds = \hat{f}(k)e_k(t)$$

and therefore the partial sums are given by convolution with certain kernels D_k. More precisely, $S_k f = f * D_k$ where

$$D_k(t) = \sum_{j=-k}^{k} e_j(t).$$

These kernels are called the Dirichlet kernels and are explicitly given by

$$D_k(t) = \frac{\sin(2k+1)\pi t}{\sin \pi t}.$$

This can be proved simply by summing a geometric series.

The Fejer means are also convolutions: $\sigma_n f = f * K_n$ where

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin(2k+1)\pi t}{\sin \pi t}.$$

Since $\sin(2k+1)\pi t$ is the imaginary part of $e^{(2k+1)\pi i t}$ another simple calculation reveals that

$$K_n(t) = \frac{1}{n+1}\frac{\sin^2(n+1)\pi t}{\sin^2 \pi t}.$$

With this explicit formula for the Fejer kernel K_n we are ready state and prove Fejers's theorem.

Theorem 1.2. *For every $f \in C(S^1), \sigma_n f$ converges to f in $C(S^1)$ as n tends to infinity.*

Proof. We first make the observation that $\int_0^1 K_n(t)dt = 1$. This follows from the fact that $\int_0^1 D_k(t)dt$ vanishes unless $k = 0$ in which case it is equal to 1. Therefore, we can write

$$\sigma_n f(t) - f(t) = \int_0^1 (f(t-s) - f(t))K_n(s)ds = \int_{-1/2}^{1/2} (f(t-s) - f(t))K_n(s)ds.$$

The equality is due to the periodicity of the functions f and K_n. As f is uniformly continuous, given $\epsilon > 0$ we can choose $\delta > 0$ such that $|f(t-s) - f(t)| < \frac{1}{2}\epsilon$ for all $|s| \leq \delta$. Therefore,

$$\int_{|s|\leq\delta} |f(t-s) - f(t)|K_n(s)ds < \frac{1}{2}\epsilon.$$

On the other hand, when $|s| > \delta, K_n(s) \to 0$ uniformly as n tends to infinity. This follows from the fact that $|\sin 2\pi s| > |\sin 2\pi\delta|$ for $\frac{1}{2} \geq |s| > \delta$. By choosing N large we can make $K_n(s) < \frac{1}{4}\|f\|_\infty^{-1}\epsilon$ for all $n \geq n$. For such n it is then immediate that

$$\int_{\delta<|s|\leq 1/2} |f(t-s) - f(t)|K_n(s)ds < \frac{1}{2}\epsilon.$$

This proves the theorem as N is independent of t. □

1.2. **Fourier series of L^2 functions.** The best behaviour of Fourier series occurs when we deal with square summable functions, i.e., functions from the Hilbert space $L^2(S^1)$. The inner product and norm in $L^2(S^1)$ are given by

$$(f,g) = \int_0^1 f(t)\bar{g}(t)dt, \ \|f\|_2^2 = (f,f)$$

for $f, g \in L^2(S^1)$. Since every $f \in L^2(S^1)$ is integrable its Fourier coefficients are well defined and we have the formal Fourier series. For the partial sums $S_n f$ of $f \in L^2(S^1)$ we have the following result.

Theorem 1.3. *For every $f \in L^2(S^1)$ the partial sums $S_n f$ converge to f in $L^2(S^1)$. Thus the exponentialse$_k, k \in \mathbb{Z}$ form an orthonormal basis for $L^2(S^1)$.*

Proof. For any trigonometric polynomial $p = \sum_{k=-m}^{m} c_k e_k$ we see that $\|p\|_2^2 = \sum_{k=-m}^{m} |c_k|^2$ due to the orthonormality of the functions e_k. Using this a simple calculation shows that

$$\|f - S_n f\|_2^2 = \|f\|_2^2 - \sum_{k=-n}^{n} |\hat{f}(k)|^2.$$

Therefore, we get the Bessel's inequality

$$\sum_{k=-n}^{n} |\hat{f}(k)|^2 \leq \|f\|_2^2$$

valid for any $f \in L^2(S^1)$. The above also shows that the series $\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$ converges which in turn implies

$$\sum_{m+1 \leq |k| \leq n} |\hat{f}(k)|^2 \to 0$$

as m tends to infinity. Since

$$\|S_n f - S_m f\|_2^2 = \sum_{m+1 \leq |k| \leq n} |\hat{f}(k)|^2$$

for m, n we see that $S_n f$ is a Cauchy sequence in $L^2(S^1)$ and hence converges to some $g \in L^2(S^1)$. The proof will be complete if we can show that $g = f$. To prove this we first observe that as S_nf converges to g in $L^2(S^1)$,

$$\hat{g}(k) = \lim_{n\to\infty} (\hat{S_n f})(k) = \hat{f}(k).$$

Thus, both f and g have the same Fourier coefficients. We just need to appeal to the following uniqueness theorem to conclude that $f = g$. □

Theorem 1.4. *For $f \in L^2(S^1)$, if $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ then $f = 0$.*

Proof. We only need to show that $\sigma_n f$ converges to f in $L^2(S^1)$ as n tends to infinity. For then under the hypothesis $\sigma_n f = 0$ for all n which means $f = 0$ as well. To show that $\sigma_n f$ converges to f in $L^2(S^1)$ we use the density of $C(S^1)$ in $L^2(S^1)$. Given $\epsilon > 0$ we choose $g \in C(S^1)$ so that $\|f-g\|_2 < \frac{1}{3}\epsilon$. Then choose N so that for all $n > N$, $\|\sigma_n g - g\|_\infty < \frac{1}{3}$ which is possible in view of Fejer's theorem. But then we also have

$$\|\sigma_n g - g\|_2 \le \|\sigma_n g - g\|_\infty < \frac{1}{3}\epsilon.$$

Therefore,

$$\|\sigma_n f - f\|_2 \le \|\sigma_n(f-g)\|_2 + \|\sigma_n g - g\|_2 + \|g - f\|_2 < \frac{2}{3}\epsilon.$$

The theorem will be proved if we can show that $\|\sigma_n f\|_2 \le \|f\|_2$ for all n. But this follows from the Young's inequality $\|f * h\|_2 \le \|h\|_1 \|f\|_2$ since $\sigma_n f = f * \sigma_n$ and $\|\sigma_n\|_1 = 1$. □

Since norm and inner product are continuos functions we obtain the following corollary, known as Parseval's theorem, as an immediate consequence of Theorem.

Corollary 1.5. *For $f, g \in L^2(S^1)$ we have*

$$(f, g) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}.$$

From the corollary we infer that the map $f \to \hat{f}(k)$ is an isometric isomorphism from $L^2(S^1)$ onto the Hilbert space of sequences $l^2(\mathbb{Z})$. This is known as the Riesz-Fischer theorem for the Hilbert space $L^2(S^1)$.

1.3. **Fourier series of L^p functions.** We now turn our attention to Fourier series of L^p functions. Equipped with the norm (for $1 \leq p < \infty$)

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}},$$

$L^p(S^1)$ is a Banach space. When $p = \infty$ we let $L^\infty(S^1)$ stand for all essentially bounded functions with $\|f\|_\infty$ being the essential supremum. Our main concern is the convergence of the partial sums $S_n f$ in the L^p norm. Instead of the partial sums if we consider the Fejer means we have the following result.

Theorem 1.6. *Let $1 \leq p < \infty$ and $f \in L^p(S^1)$. Then $\sigma_n f$ converges to f in the norm.*

Proof. Since $C(S^1)$ is dense in $L^p(S^1)$ the theorem will follow if we can show: (i) $\sigma_n g$ converges to g in $L^p(S^1)$ for all $g \in C(S^1)$;(ii) $\|\sigma_n f\|_p \leq \|f\|_p$, for all $f \in L^p(S^1)$ where C is independent of n. (We say that σ_n are uniformly bounded on $L^p(S^1)$.) To see this, let ϵ be given. First choose $g \in C(S^1)$ such that $\|f - g\|_p < \frac{1}{4}\epsilon$ and then take N so that $\|\sigma_n g - g\|_p < \frac{1}{2}\epsilon$ for all $n \geq N$. Then it is clear that

$$\|\sigma_n f - f\|_p \leq \|\sigma_n(f - g)\|_p + \|\sigma_n g - g\|_p + \|f - g\|_p < \epsilon$$

for all $n \geq N$.

The assertion (i) follows from Fejer's theorem since the uniform norm dominates the L^p norm and (ii) follows from Young's inequality. This completes the proof of the theorem. □

The above theorem shows that the set of all trigonometric polynomials is dense in L^p for $1 \leq p < \infty$ (which is clearly not true for $L^\infty(S^1)$). When f is a trigonometric polynomial it is clear that $S_n f$ converges to f in the norm as n tends to infinity. Therefore, in order to show that $S_n f$ converges to f in the norm for all $f \in L^p(S^1)$ we only need to prove that the operators S_n are uniformly bounded on L^p. In fact, by applealing to uniform boundedness principle we can show that these two are equivalent. Regarding the uniform boundedness of S_n we begin with the following negative result.

Theorem 1.7. *The operators S_n are not uniformly bounded on $L^1(S^1)$. Consequently there are integrable functions f for which $S_n f$ does not converge in $L^1(S^1)$.*

Proof. We prove the theorem by contradiction. Suppose we have the uniform estimates $\|S_n f\|_1 \leq C\|f\|_1$ for some constant $C > 0$. Taking $f = \sigma_m$, the Fejer kernel we see that $\|D_n * \sigma_m\|_1 \leq C$ for all n and m. Since $D_n * \sigma_m$ converges to D_n in $L^1(S^1)$ as m tends to infinity we get $\|D_n\|_1 \leq C$ for all n. But an easy computation, which we leave it to the reader, shows that $\|D_n\|_1$ grows like $logn$. Hence, we get the required contradiction. □

Let us look at $S_n f$ more closely. We can rewrite

$$S_n f(t) = e_n(-t) \sum_{k=0}^{2n} \hat{f}(k-n) e_k(t)$$

and since $\hat{f}(k-n) = (\widehat{fe_n})(k)$ we have the equation $e_n S_n f = P_{2n}(fe_n)$ where $P_n f = \sum_{k=0}^{n} \hat{f}(k-n)e_k$. Thus the uniform boundedness of S_n will follow once we prove the uniform boundedness of P_n. These operators suggest that we look at the projection operator P defined by

$$Pf = \sum_{k=0}^{\infty} \hat{f}(k)e_k.$$

We observe that Pf is defined on $C(S^1)$ as an L^2 function. A priori it is not clear if we can extend P to $L^p(S^1)$ as a bounded linear operator.

Proposition 1.8. *The partial sum operators S_n are uniformly bounded on $L^p(S^1)$ if and only if P initially defined on $C(S^1)$ extends to $L^p(S^1)$ as a bounded operator.*

Proof. If S_n are uniformly bounded then so are P_n and hence we have $\|P_n f\|_p \leq C\|f\|_p$ for all $f \in L^p(S^1)$ and C independent of n. For any trigonometric polynomial f of degree m we see that $P_n f = Pf$ for all $n \geq m$ and hence $\|Pf\|_p \leq C\|f\|_p$. Since trigonometric polynomilas are dense in $L^p(S^1)$ we can extend P to the whole of L^p as a bounded operator. Conversely, suppose P is bounded on $L^p(S^1)$. If f is a trigonometric polynomial of degree m then $\|P_n f\|_p = \|Pf\|_p \leq C\|f\|_p$ for all $n \geq m$ and hence $\|P_n f\|_p \leq C(f)$ for all n for some $C(f)$. By appealing to the uniform boundedness principle we get $\|P_n\|_p \leq C\|f\|_p$ and the same is true of S_n. This proves the proposition. □

Thus we have reduced the problem of proving the uniform boundedness of S_n to that of proving the boundedness of the single operator P. When $f \in C(S^1)$ the boundedness of the Fourier coefficients allow us to extend Pf as a holomorphic function of $z = re^{2\pi it}$ in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Indeed,

$$Pf(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$$

is holomorphic in D as the series converges uniformly over compact subsets of D. Note that P is a convolution operator given by

$$Pf(re^{2\pi it}) = \int_0^1 f(t-s)(1 - re^{2\pi is})^{-1} ds.$$

When f is real valued a simple calculation shows that

$$2Pf(re^{2\pi it}) = \hat{f}(0) + P_r f(t) + iQ_r f(t)$$

where

$$P_r f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e_k(t)$$

and

$$Q_r f(t) = (-i) \sum_{k=-\infty}^{\infty} \operatorname{sign}(k) \hat{f}(k) r^{|k|} e_k(t).$$

Here $\operatorname{sign}(k) = 1$ if $k > 0, -1$ if $k < 0$ and $\operatorname{sign}(0) = 0$.

The functions $P_r f$ and $Q_r f$ are called the Poisson and conjugate Poisson integrals of f respectively. Observe that $Pf(re^{2\pi it})$ converges to $Pf(e^{2\pi it})$ as $r \to 1$ whenever f is a trigonometric polynomial. Therefore, the inequality $\|Pf\|_p \leq C\|f\|_p$ for all trigonometric polynomials will follow if we can show that P_r and Q_r are uniformly bounded on $L^p(S^1)$ for all $0 < r < 1$.

We first prove the uniform boundedness of P_rf which is easy. We can write $P_rf(t) = f * p_r(t)$ where the 'Poisson kernel' $p_r(t)$ is given by

$$p_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e_k(t) = \Re\left((1 - re^{2\pi is})^{-1}\right) - 1.$$

A simple calculation shows that

$$p_r(t) = \frac{1 - r^2}{1 + r^2 - 2r\cos 2\pi t}.$$

Note that $p_r(t) > 0$ and $\|p_r\|_1 = 1$. Therefore, the following theorem is an immediate consequence of Young's inequality.

Theorem 1.9. *For $1 \leq p < \infty, f \in L^p(S^1)$ and $0 < r < 1$ we have the uniform estimates $\|P_rf\|_p \leq C\|f\|_p$. Consequently, P_rf converges to f in the norm as r tends to one.*

The uniform boundedness of Q_rf is not so easy to establish. As above we can write $Q_rf(t) = f * q_r(t)$ where the 'conjugate Poisson' kernel is given by

$$q_r(t) = (-i) \sum_{k=-\infty}^{\infty} (\operatorname{sign} k) r^{|k|} e_k(t) = \Im\left((1 - re^{2\pi is})^{-1}\right).$$

More explicitly,

$$q_r(t) = \frac{2r\sin 2\pi t}{1 + r^2 - 2r\cos 2\pi t}.$$

We observe that unlike $p_r(t)$, this kernel is oscillating and $\int_0^1 q_r(t)dt = 0$. We therefore, cannot use the simple-minded Young's inequality in proving the uniform boundedness of the conjugate Poisson integrals. The idea is to use the fact that $P_rf(t) + iQ_rf(t)$ is a holomorphic function of $z = re^{2\pi it}$. We show that the L^p norms of Q_rf can be estimated in terms of the L^p norms of P_rf. By the result of the previous theorem we get the uniform boundedness of Q_rf.

Theorem 1.10. *For $1 < p < \infty, f \in C(S^1)$ and $0 < r < 1$ we have the uniform estimates $\|Q_rf\|_p \leq C\|f\|_p$. Consequently, Q_rf converges to some function $\tilde{f}$ (called the conjugate function) in the norm as r tends to one.*

Proof. Without loss of generality we can assume that $f \geq 0$ so that the real part of the holomorphic function $F(z) = P_r f(t) + iQ_r f(t)$ is positive and hence $G(z) = F(z)^p$ is welldefined and holomorphic in D. We can just choose the branch which is real at the origin. Let γ be such that $\gamma < \frac{\pi}{2}$ but $p\gamma > \frac{\pi}{2}$ and define $A_\gamma = \{t \in [0,1) : |\arg F(re^{2\pi it})| < \gamma\}$ and B_γ to be the complement of A_γ in $[0,1)$. We first estimate the integral of $|G(z)|$ taken over A_γ.

Since $F(re^{2\pi it}) = |F(re^{2\pi it})|e^{i\arg F(re^{2\pi it})}$ we have

$$\Re F(re^{2\pi it}) = |F(re^{2\pi it})|\cos(\arg F(re^{2\pi it})) \geq |F(re^{2\pi it})\cos\gamma$$

for all $t \in A_\gamma$. Therefore,

$$\int_{A_\gamma} |F(re^{2\pi it})|^p dt \leq (\cos\gamma)^{-p} \int_{A_\gamma} P_r f(t)^p dt \leq C_p \int_0^1 |f(t)|^p dt$$

where we have used the uniform boundedness of the Poisson integrals.

On the other hand,

$$\Re G(re^{2\pi it}) = |F(re^{2\pi it})|^p \cos(p \arg F(re^{2\pi it}))$$

and hence for $t \in B_\gamma$ we have

$$|G(re^{2\pi it})| \leq (\cos(p\gamma))^{-1} \Re G(re^{2\pi it}).$$

Mean value theorem applied to the harmonic function $\Re G(re^{2\pi it})$ gives us

$$\int_0^1 \Re G(re^{2\pi it}) dt = \hat{f}(0)$$

from which we get

$$\int_{B_\gamma} \Re G(re^{2\pi it}) dt = \hat{f}(0) - \int_{A_\gamma} \Re G(re^{2\pi it}) dt.$$

As both terms on the right hand side are uniformly bounded by $\|f\|_p^p$ we get

$$\int_{B_\gamma} |G(re^{2\pi it})| dt \leq |\cos(p\gamma)|^{-1} |\int_{B_\gamma} \Re G(re^{2\pi it}) dt| \leq C_p \int_0^1 |f(t)|^p dt.$$

Thus we have proved

$$\int_0^1 |F(re^{2\pi it})|^p dt \leq C_p \int_0^1 |f(t)|^p dt.$$

As $Q_rf(t) = \Im F(re^{2\pi it})$ this proves the uniform boundedness of the conjugate Poisson integrals.

□

As noted earlier the uniform boundedness of the Poisson and conjugate Poisson integrals lead to the boundedness of the projection operator P on L^p for $1 < p < \infty$. This together with Proposition gives the following result.

Theorem 1.11. *The partial sum operators S_n are uniformly bounded on L^p for all $1 < p < \infty$. Consequently, for every $f \in L^p(S^1)$, S_nf converges to f in the norm as n tends to infinity.*

We conclude this subsection with some remarks on the conjugate function $\tilde{f}$ which we mentioned in passing in the statement of Theorem . For $f \in L^2(S^1)$ the series

$$(-i) \sum_{k=-\infty}^{\infty} (\operatorname{sign} k)\hat{f}(k)e_k$$

converges in L^2 to a function which is denoted by $\tilde{f}$. Since $\|\tilde{f}\|_2 \leq \|f\|_2$ the operator $f \to \tilde{f}$ is bounded on $L^2(S^1)$. For trigonometric polynomials, Q_rf converges to $\tilde{f}$ as $r \to 1$. Therefore, the uniform boundedness of Q_rf leads to the estimate $\|\tilde{f}\|_p \leq C\|f\|_p$ for all $1 < p < \infty$. Since $Q_rf = f * q_r$ we expect $\tilde{f}$ to be given by

$$\tilde{f}(t) = \int_0^1 f(s) \cot(t-s) ds.$$

In the above representation the kernel $\cot(t - s)$ has a nonintegrable singularity along the diagonal and hence the above is a 'singular integral operator' and dealing with them is a delicate problem.

1.4. **Fourier series and holomorphic functions.** In connection with the operator P we have considered functions on S^1 which have a holomorphic extension to the disc D. Any $f \in C(S^1)$ for which $\hat{f}(k) = 0$ for all $k < 0$ has this property. For such functions $f = Pf$ so that

$$2f(re^{2\pi it}) = \hat{f}(0) + P_rf(t) + iQ_rf(t)$$

from which we get the uniform estimates

$$\int_0^1 |f(re^{2\pi it})|^p dt \leq C_p^p$$

for all $1 < p < \infty$ $0 < r < 1$. This leads to the definition of the (holomorphic) Hardy spaces $H^p(D)$. In this section we consider functions on S^1 which have holomorphic extensions to $\mathbb{C}^*$, the set of all nonzero complex numbers.

As we know any function $F(z)$ holomorphic in an annulus $r_1 < |z| < r_2$ has the Laurent expansion

$$F(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

the series being uniformly convergent on every compact subset of the annulus. When $|z| = r, r_1 < r < r_2$ the above expansion is nothing but the Fourier series of the function $F(re^{2\pi it})$. Whenever the annulus contains S^1 this simply means that $a_k = \hat{f}(k)$, where f is the restriction of F to S^1. Writing $z = e^{2\pi i(t+is)}$ where $s \in \mathbb{R}$ is such that $r_1 < e^{-2\pi s} < r_2$ the above series takes the form

$$F(e_1(t+is)) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi ik(t+is)}.$$

Applying Parseval's theorem we get the following formula known as Gutzmer's formula in the literature.

Lemma 1.12. *Let F be holomrphic in an annulus which contains S^1 and let f be the restriction of F to S^1. Then for any $s \in \mathbb{R}$ for which $e^{2\pi i(t+is)}$ is in the annulus we have*

$$\int_0^1 |F(e_1(t+is))|^2 dt = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 e^{-4\pi ks}.$$

When F is holomorphic on $\mathbb{C}^*$ the above formula is valid for all $s \in \mathbb{R}$. One way to produce such functions is to start with a function f on S^1 and define

$$F(e_1(t+is)) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi ik(t+is)}.$$

For the series to converge and define a holomorphic function we certainly need to assume that the Fourier coefficients $\hat{f}(k)$ have rapid decay. By fixing a function h with rapdly decreasing Fourier coeffients and considering $f * h$ in place of f where f runs through, say $L^2(S^1)$ we get a whole family of functions with the desired property.

Let us be more specific and consider the function $h_r, r > 0$ defined by the condition $\hat{h_r}(k) = e^{-4\pi^2 rk^2}$. The function itself is given by the uniformly convergent series

$$h_r(t) = \sum_{k=-\infty}^{\infty} e^{-4\pi^2 rk^2} e^{2\pi ikt}.$$

It is clear that h_r extends to a holomorphic function on $\mathbb{C}^*$. For any $f \in L^2(S^1)$ the function $u_r = f * h_r$ also has the same property due to the fact that $f \hat{*} h_r(k) = \hat{f}(k)\hat{h_r}(k)$. From the definition we observe that $u_r(t)$ satisfies the heat equation

$$\partial_r u_r(t) = \partial_t^2 u_r, \ \lim_{r\to 0} u_r(t) = f(t)$$

where the limit is taken in the L^2 sense. Indeed, we have

Theorem 1.13. *For any $f \in L^p(S^1), 1 \leq p < \infty, f * h_r$ converges to f in the norm as $r \to 0$.*

Proof. We show below that $h_r(t) > 0, \|h_r\|_1 = 1$ and $h_r(t) \to 0$ uniformly for $\delta < |t| \leq 1$ as r tends to 0. The first two properties give us the uniform boundedness $\|f * h_r\|_p \leq \|f\|_p$. The third property shows that $f * h_r$ converges to f uniformly for all $f \in C(S^1)$. These two will complete the proof. The required properties of h_r follow from the next result known as Jacobi's identity. □

Proposition 1.14. *For any $r > 0$ and $t \in \mathbb{R}$ we have*

$$h_r(t) = (4\pi r)^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{4r}(t-k)^2}.$$

Proof. The series on the right hand side converges uniformly and defines a smooth periodic function. Therefore, we only need to show that the Fourier coefficients of that function are precisely $e^{-4\pi^2 rk^2}$. Changing the order of integration and summation we see that the $k-$th Fourier coefficient is given by

$$(4\pi r)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4r}t^2} e^{-2\pi ikt}.$$

This integral is evaluated in the next lemma which proves the proposition. □

Lemma 1.15.

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} e^{-its} dt = e^{-\frac{1}{2}s^2}.$$

Proof. The integral clearly defines an entire function of the complex variable s and hence it is enough to show that

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} e^{ts} dt = e^{\frac{1}{2}s^2}.$$

By changing t into $t+s$ the integral under consideration becomes

$$(2\pi)^{-\frac{1}{2}} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt \right) e^{\frac{1}{2}s^2}.$$

This proves the lemma as the last integral is $(2\pi)^{-\frac{1}{2}}$. □

We have shown that the functions $u_r(t) = f * h_r(t), f \in L^2(S^1)$ extend to $\mathbb{C}^*$ as holomorphic functions. They also have another interesting property. Let us define a weight function $w_r(z)$ on $\mathbb{C}^*$ by setting

$$w_r(z) = (2\pi r)^{-\frac{1}{2}} e^{-\frac{1}{2r}s^2}, \ z = e^{2\pi i(t+is)}.$$

A simple calculation shows that

$$(2\pi r)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-4\pi ks} e^{-\frac{1}{2r}s^2} ds = e^{8\pi^2 rk^2}.$$

In view of Gutzmer's formula and Parsevals' theorem we obtain

$$\int_{\mathbb{C}^*} |f * h_r(z)|^2 w_r(z) dz = \int_0^1 |f(t)|^2 dt.$$

Here $dz = dtds$ is the measure on $\mathbb{C}^*$ which is identified with $S^1 \times \mathbb{R}$. Thus the holomorphic extension of $f * h_r$ belongs to the weighted

Bergman space $\mathcal{B}_r(\mathbb{C}^*)$ defined to be the space of all holomorphic functions on $\mathbb{C}^*$ that are square integrable with respect to $w_r(z)dz$. We have the follwoing characterisation of this space.

Theorem 1.16. *A holomorphic function F on $\mathbb{C}^*$ belongs to $\mathcal{B}_r(\mathbb{C}^*)$ if and only if the restriction of F to S^1 is of the form $f * h_r$ for some $f \in L^2(S^1)$. Moreover,*

$$\int_{\mathbb{C}^*} |F(z)|^2 w_r(z) dz = \int_0^1 |f(t)|^2 dt.$$

Proof. We only need to prove the 'only if' part as the other part has been proved above. By Gutzmer's formula we have

$$\int_0^1 |F(e_1(t+is))|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 e^{-4\pi ks}$$

where c_k are the Fourier coefficients of the function $g(t) = F(e_1(t))$. Integrating with respect to $(2\pi r)^{-\frac{1}{2}} e^{-\frac{1}{2r}s^2} ds$ and using the hypothesis on F we see that

$$\sum_{k=-\infty}^{\infty} |c_k|^2 e^{8\pi^2 rk^2} < \infty.$$

In order to complete the proof we just define f by

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{4\pi rk^2} e_k(t)$$

so that $f \in L^2(S^1)$ and $f * h_r(t) = c_k = \hat{g}(k)$ as desired. □

The above theorem shows that the taranform which takes f into the holomorphic function $F(z) = f * h_r(e_1(t+is))$ is an isometric isomorphism from $L^2(S^1)$ onto $\mathcal{B}_r(\mathbb{C}^*)$. This transform is called the Segal-Bargmann transform and also the heat kernel transform which can be studied in various other settings as well.

2. Fourier transform on the real line

In this section we show that there exists a remarkable unitary operator $\mathcal{F}$ on the Hilbert space $L^2(\mathbb{R})$ which we call the Fourier transform and study some of the basic properties of that operator.

2.1. Unitary operators: some examples.

We begin with some definitions. Given two Hilbert spaces H_1 and H_2 consider a bounded linear operator $T : H_1 \to H_2$. We define its adjoint, denoted by T^* the unique operator from H_2 into H_1 determined by the condition

$$(Tu, v)_2 = (u, T^*v)_1, \ u \in H_1, v \in H_2$$

where $(\cdot, \cdot)_j$ stand for the inner product in H_j. Note that T^* is bounded. We say that T is unitary if $TT^* = I_2, T^*T = I_1$ where I_j is the identity operator on H_j. If T is unitary then we have $(u, v)_1 = (Tu, Tv)_2$ for all $u, v \in H_1$. In particular $\|Tu\|_2 = \|u\|_1, u \in H_1$.

We give some examples of unitary operators. Let $H_1 = L^2(S^1)$ and $H_2 = l^2(\mathbb{Z})$. Take T to be the operator $Tf(k) = \hat{f}(k)$ where

$$\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}dt.$$

Then it can be checked that T^* is given by

$$T^*\varphi(t) = \sum_{-\infty}^{\infty} \varphi(k)e^{-2\pi ikt}.$$

The Plancherel theorem for the Fourier series shows that T is unitary. Another simple example is provided by the translation $\tau_a f(x) = f(x - a)$ defined from $L^2(\mathbb{R})$ into itself. We give some more examples below.

Let us take the nonabelian group $\mathbb{H}^1$ which is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + xy').$$

Then it is clear that $\mathbb{H}^1$ is nonabelian and the Lebesgue measure $dxdydt$ is both left and right invariant Haar measure on $\mathbb{H}^1$. With this measure we can form the Hilbert space $L^2(\mathbb{H}^1)$. Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then it is easy to check that Γ is a subgroup of $\mathbb{H}^1$ so that we can form the quotient $M = \Gamma/\mathbb{H}^1$ consisting of all right cosets of Γ. Functions on M are naturally identified with left $\Gamma-$invariant functions on $\mathbb{H}^1$. As the Lebesgue measure $dxdydt$ is left $\Gamma-$ invariant we can form $L^2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0,1)^3$ and we just think of $L^2(M)$ as $L^2([0,1)^3)$.

Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x,y,t+s) = e^{2\pi iks} f(x,y,t).$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k, \;\; f_k \in \mathcal{H}_k.$$

We are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right.(why?) It is interesting to note that functions in $\mathcal{H}_1$ are also invariant under the left action of Γ.

Our next example of a unitary operator is the following. Consider the map $J : \mathbb{H}^1 \to \mathbb{H}^1$ given by $J(x,y,t) = (y,-x,t-xy)$. Then J is an automorphism of the group $\mathbb{H}^1$ which satisfies (i)$J^4 = I$,(ii) $J(\Gamma) = \Gamma$(i.e. J leaves Γ invariant) and (iii) J restricted to the center of $\mathbb{H}^1$ is just the identity; i.e. $J(0,0,t) = (0,0,t)$. Using this automorphism we define an operator, denoted by the same symbol, on $\mathcal{H}_1$ by

$$Jf(x,y,t) = f(J(x,y,t)) = f(y,-x,t-xy).$$

It is clear that $J^*f(x,y,t) = f(-y,x,t-xy)$ so that J is unitary. We also observe that $J^2f(x,y,t) = f(-x,-y,t)$.

We now define another very important unitary operator which takes $L^2(\mathbb{R})$ onto $\mathcal{H}_1$. This operator used by Weil and Brezin is called the Weil- Brezin transform and is defined as follows. For $f \in L^2(\mathbb{R})$,

$$Vf(x,y,t) = e^{2\pi it} \sum_{n=-\infty}^{\infty} f(x+n)e^{2\pi iny}.$$

As $f \in L^2(\mathbb{R})$ we know that $f(x+n)$ is finite for almost every $x \in \mathbb{R}$. The above series converges in $L^2([0,1))$ as a function of y and we have

$$\int_0^1 |Vf(x,y,t)|^2 dy = \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)|^2.$$

Thus it follows that $Vf \in \mathcal{H}_1$ and

$$\int_{[0,1)^3} |Vf(x,y,t)|^2 dxdydt = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Proposition 2.1. *V is a unitary operator from $L^2(\mathbb{R})$ onto $\mathcal{H}_1$.*

To prove this proposition we need to calculate V^*. It is clear that V is one to one but is also onto. To see this, given $F \in \mathcal{H}_1$ consider f defined as follows. For $x \in [m, m+1)$ define

$$f(x) = \int_0^1 F(x-m, y, 0) e^{-2\pi i m y} dy.$$

Then it is clear that $f \in L^2(\mathbb{R})$ and

$$Vf(x,y,t) = e^{2\pi i t} \sum_{m=-\infty}^{\infty} \left(\int_0^1 F(x,u,0)) e^{-2\pi i m u} du \right) e^{2\pi i m y} = F(x,y,t).$$

Moreover, if $f, g \in L^2(\mathbb{R})$ then

$$(f,g) = \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m) \overline{g(x+m)} dx.$$

The sum is nothing but

$$\int_0^1 Vf(x,y,t) \overline{Vg(x,y,t)} dy$$

and hence we have $(f,g) = (Vf, Vg)$. This shows that $V^* = V^{-1}$ and hence V is unitary.

2.2. **Fourier transform: Plancherel and inversion theorems.**

Definition 2.2. *The unitary operator V^*JV from $L^2(\mathbb{R})$ onto itself is called the Fourier transform and is denoted by $\mathcal{F}$.*

We record some important properties of the Fourier transform in the following theorem.

Theorem 2.3. *The Fourier transform $\mathcal{F}$ satisfies: (i) $\mathcal{F}^4 f = f$, for every $f \in L^2(\mathbb{R})$ (ii) $\mathcal{F}^2 f(x) = f(-x)$ for almost every $x \in \mathbb{R}$ and (iii) $\|\mathcal{F}f\|_2 = \|f\|_2$.*

We only need to check (ii) as (i) follows immediately since $J^4 = I$. As $J^2 f(x, y, t) = f(-x, -y, t)$ we have

$$\mathcal{F}^2 f(x) = \int_0^1 V f(-x - m, -y, t) e^{-2\pi i m y} dy$$

whenever $x \in [m, m+1)$. If we recall the definition of Vf the above is simply $f(-x)$.

The property (iii), namely $\|\mathcal{F}f\|_2 = \|f\|_2$ is called the Plancherel theorem for the Fourier transform.

Before proceeding further let us calculate the Fourier transforms of some well known functions. As our first example let us take the Gaussian $\varphi(x) = e^{-\pi x^2}$.

Proposition 2.4. *The Fourier transform of φ is itself: $\mathcal{F}\varphi = \varphi$.*

Proof. By definition, when $x \in [0, 1)$,

$$\mathcal{F}\varphi(x + m) = \int_0^1 \sum_{n=-\infty}^{\infty} \varphi(y + n) e^{-2\pi i (y+n)(x+m)} dy$$

which can be rewritten as

$$e^{-\pi(x+m)^2} \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(y+n+i(x+m))^2} dy.$$

We claim that the integral is a constant. To see this, note that

$$G(w) = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(y+n+iw)^2} dy = \int_{-\infty}^{\infty} e^{-\pi(y+iw)^2} dy$$

is an entire function of $w = u + iv$ and $G(iv) = G(v)$. Hence $G(x + m)$ is a constant and we get $\mathcal{F}\varphi = c\varphi$. But $G(0) = 1$ and so $c = 1$ proving the proposition. □

The above proposition shows that the Gaussian φ is an eigenfunction of the Fourier transform. We will say more about the spectral decomposition of $\mathcal{F}$ in the next section.

We introduced the Fourier transform as a unitary operator on $L^2(\mathbb{R})$. Now we extend the definition to $L^1(\mathbb{R})$ and prove a useful inversion formula.

Theorem 2.5. *For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the Fourier transform is given by*

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

If we further assume that $\mathcal{F}f \in L^1(\mathbb{R})$ then for almost every x we have

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(\xi) e^{2\pi i x\xi} d\xi.$$

Proof. If $\xi = x + m, x \in [0, 1)$ it follows from the definition that

$$\mathcal{F}f(\xi) = \int_0^1 \sum_{n=-\infty}^{\infty} f(y+n) e^{-2\pi i (y+n)(x+m)} dy.$$

As f is integrable we can interchange the order of summation and integration to arrive at the formula

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

Under the assumption that $\mathcal{F}f$ is also integrable the inversion formula $\mathcal{F}^2 f(x) = f(-x)$ leads to

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(\xi) e^{2\pi i x\xi} d\xi.$$

This completes the proof of the theorem. □

It is customary to denote the Fourier transform $\mathcal{F}f$ of integrable functions by $\hat{f}$. Thus

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

It is clear from this that the Fourier transform can be defined on all of $L^1(\mathbb{R})$. Note that $\hat{f}$ for $f \in L^1(\mathbb{R})$ is a bounded function and

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)| dx = \|f\|_1.$$

It can be easily checked, by an application of the Lebesgue dominated convergence theorem, that $\hat{f}$ is in fact continuous. But something more is true. The following result is known as the Riemann-Lebesgue lemma in the literature.

Theorem 2.6. *For all* $f \in L^1(\mathbb{R}), \hat{f}$ *vanishes at infinity; i.e.,* $\hat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$.

Proof. As $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ it is enough to prove the result for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that $\hat{f}(x+m), x \in [0,1)$ is the $m-$th Fourier coefficient of the integrable periodic function

$$F(y) = \sum_{n=-\infty}^{\infty} f(y+n) e^{-2\pi i x(y+n)}$$

and hence it is enough to show that the Fourier coefficients of an integrable function vanish at infinity. It is clearly true of trigonometric polynomials and as they are dense in $L^1([0,1))$ the same true for all integrable functions. □

Another immediate consequence of our definition of the Fourier transform is the so called Poisson summation formula. If the integrable function f satisfies the estimate $|f(y)| \leq C(1+y^2)^{-1}$ then the series defining $Vf(x,y,t)$ converges uniformly. The same is true of $V\hat{f}$ if $\hat{f}$ also satisfies such an estimate. For such functions we have the following result.

Theorem 2.7. *Assume that* f *is measurable and satisfies* $|f(y)| \leq C(1+y^2)^{-1}$ *and* $|\hat{f}(\xi)| \leq C(1+\xi^2)^{-1}$. *Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. Since $\hat{f} = V^*JVf$ we have $JVf(x,y,t) = V\hat{f}(x,y,t)$. As both series defining JVf and $V\hat{f}$ converge uniformly we can evaluate them at $(0,0,0)$ which gives the desired result. □

When we take $f(x) = t^{-\frac{1}{2}}\varphi(t^{-\frac{1}{2}}x)$ for $t > 0$, it follows that $\hat{f}(\xi) = \varphi(t^{\frac{1}{2}}\xi)$ and hence Poisson summation formula gives the interesting identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = t^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\pi t^{-1} n^2}.$$

We can obtain several identities of this kind by considering eigenfunctions of the Fourier transform.

2.3. **Spectral decomposition of $\mathcal{F}$.** The spectrum of the Fourier transform $\mathcal{F}$ is contained in the unit circle as $\mathcal{F}$ is unitary. Moreover, as $\mathcal{F}^4 = I$ any λ in its spectrum $\sigma(\mathcal{F})$ satisfies $\lambda^4 = 1$. Hence, $\sigma(\mathcal{F}) = \{1, -1, i, -i\}$. In this subsection we describe explicitly the orthogonal projections associated to each point of the spectrum.

We have identified at least one eigenfunction, namely the Gaussian. Let us search for eigenfunctions of the form $f(x) = p(x)e^{-\pi x^2}$ where p is a real valued polynomial. The reason is the following: the Fourier transform of such a function is given by,

$$\mathcal{F}f(x+m) = e^{-\pi(x+m)^2} \int_0^1 \sum_{n=-\infty}^{\infty} p(y+n) e^{-\pi(y+n+i(x+m))^2} dy$$

for $x \in [0, 1)$. As before we are led to consider the function

$$G(w) = \int_{-\infty}^{\infty} p(y) e^{-\pi(y+iw)^2} dy$$

which is entire. For $v \in \mathbb{R}$,

$$G(iv) = \int_{-\infty}^{\infty} p(y+v) e^{-\pi y^2} dy$$

is a polynomial and hence $\mathcal{F}f(x) = q(x)e^{-\pi x^2}$ for another polynomial q. So it is reasonable to expect eigenfunctions among this class of functions. Let us record this in the following.

Proposition 2.8. *Let p be a polynomial with real coefficients. Then $f(x) = p(x)e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue λ if and* *only if*

$$\int_{-\infty}^{\infty} p(x-iy)e^{-\pi x^2}dx = \lambda p(y).$$

From the above equation we can infer several things. Calculating the the derivatives at the origin we have

$$(-i)^k \int_{-\infty}^{\infty} p^{(k)}(x)e^{-\pi x^2}dx = \lambda p^{(k)}(0).$$

Since p is real valued and $\lambda = (-i)^n$ for $n = 0, 1, 2, 3$ the degree m of p should satisfy the condition $(-i)^{m+n}$ is real. This means m should be odd (even) whenever n is odd (resp. even). Moreover, since $\mathcal{F}^2 f(x) = f(-x)$ we infer that the polynomials p corresponding to real (imaginary) eigenvalues are even (resp. odd) functions. If we assume that each p is monic then we also get that p should be of degree $4k+n$ if $\lambda = (-i)^n, n = 0, 1, 2, 3$. With these preparations we can easily show the existence of eigenfunctions of the Fourier transform.

Theorem 2.9. *There exist monic polynomials p_k of degree $k \in \mathbb{N}$ such that $p_k(x)e^{-\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue $(-i)^k$.*

Proof. We consider only the case of $\lambda = 1$. The other cases can be treated similarly. In this case we have to find polynomials p of degree $4k$ such that

$$\int_{-\infty}^{\infty} p(x-iy)e^{-\pi x^2}dx = p(y).$$

This leads to the equations

$$\sum_{j=0}^{4k} \frac{1}{j!} C_j (-iy)^j = \sum_{j=0}^{4k} \frac{1}{j!} p^{(j)}(0) y^j$$

where

$$C_j = \int_{-\infty}^{\infty} p^{(j)}(x)e^{-\pi x^2}dx.$$

As p is a function of x^2 it follows that $C_j = 0$ unless j is even. Thus we are led to the equations

$$p^{(2j)}(0) = (-1)^j \int_{-\infty}^{\infty} p^{(2j)}(x)e^{-\pi x^2}dx.$$

These equations can be solved recursively starting with $p^{(4k)}(0) = (4k)!$. The details are left to the reader. $\square$

The polynomials whose existence is guaranteed by the above theorem are called the Hermite polynomials and denoted by $H_k(x)$. We define the Hermite functions $\varphi_k(x) = c_k H_k(x) e^{-\pi x^2}$ with suitably chosen c_k so as to make $\|\varphi_k\|_2 = 1$. The importance of the Hermite functions lie in the following theorem.

Theorem 2.10. *The Hermite functions* $\varphi_k, k \in \mathbb{N}$ *form an orthonormal basis for* $L^2(\mathbb{R})$.

Proof. Here we only prove that they form an orthonormal system. The completeness will be proved later. Since $(f, g) = (\hat{f}, \hat{g})$ for all $f, g \in L^2(\mathbb{R})$ it follows that

$$(\varphi_{4k+n}, \varphi_{4j+m}) = (-i)^{n-m}(\varphi_{4k+n}, \varphi_{4j+m})$$

for any $n, m \in \{0, 1, 2, 3\}$. Thus $(\varphi_{4k+n}, \varphi_{4j+m}) = 0$ whenever $n \neq m$. This argument does not prove the orthogonality within the same eigenspace.

Consider the operator $H = -\frac{d^2}{dx^2} + 4\pi^2 x^2$. By integration by parts we can easily verify that $\mathcal{F}(Hf) = H(\mathcal{F}f)$ for all functions of the form $f(x) = p(x)e^{-\pi x^2}$ with p polynomial. The above shows that $H\varphi_k$ is an eigenfunction of $\mathcal{F}$ with the same eigenvalue and hence there are real constants λ_k such that $H\varphi_k = \lambda_k \varphi_k$. Integration by parts also shows that $(Hf, g) = (f, Hg)$ which leads to the equation

$$\lambda_k(\varphi_k, \varphi_j) = \lambda_j(\varphi_k, \varphi_j).$$

If we can show that $\lambda_k \neq \lambda_j$ for $k \neq j$ then we can conclude that

$$(\varphi_k, \varphi_j) = \delta_{jk} \int_{\mathbb{R}} \varphi_k(x)^2 dx$$

. To prove that the λ_k are distinct, we observe that the equation $Hf = \lambda_k f$ for $f(x) = p(x)e^{-\pi x^2}$ reduces to

$$-p^{(2)}(x) + 4\pi x p^{(1)}(x) + 2\pi p(x) = \lambda_k p(x).$$

If p is a polynomial of degree k, a comparison of the coefficients of x^k on both sides of the above equation shows that $\lambda_k = 2\pi(2k+1)$. This proves our claim. □

We are now ready to state the explicit spectral decomposition of $\mathcal{F}$. For $j = 0, 1, 2, 3$ define $L^2_j(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ for which $\{\varphi_{4k+j} : k \in \mathbb{N}\}$ is an orthonormal basis. Let P_j stand for the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2_j(\mathbb{R})$.

Theorem 2.11. *We have* $L^2(\mathbb{R}) = \oplus_{j=0}^3 L^2_j(\mathbb{R})$. *Every* $f \in L^2(\mathbb{R})$ *can be uniquely written as* $f = \sum_{j=0}^3 P_j f$. *The projections are explicitly given by the Hermite expansion*

$$P_j f = \sum_{k=0}^{\infty} (f, \varphi_{4k+j})\varphi_{4k+j}.$$

We conclude this section with the following generalisation of the Poisson summation formula.

Theorem 2.12. *Let* $f = \varphi_{4k+j}$ *be any of the Hermite functions. Then we have*

$$\sum_{n=-\infty}^{\infty} f(y+n)e^{-2\pi i x(y+n)} = (-i)^j \sum_{n=-\infty}^{\infty} f(x+n)e^{2\pi i n y}.$$

Proof. As $\mathcal{F} = V^* J V$ the equation $\mathcal{F}f = (-i)^j f$ translates into $JVf(x,y,t) = (-i)^j Vf(x,y,t)$. This proves the theorem. □

2.4. **Theta transform and Hardy's theorem.** In this section we return to the Hilbert space $L^2(M)$ introduced in section 1.1. We introduce and study a transform called the theta transform. As applications we show that the Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$ and prove a theorem of Hardy.

Let $\varphi_{i\tau}(x) = e^{\pi i \tau x^2}$ which belongs to $L^2(\mathbb{R})$ even for complex τ provided $\Im(\tau) > 0$. Let $\psi_{i\tau}(x) = \frac{1}{2\pi i \tau}\frac{\partial}{\partial x}\varphi_{i\tau}(x) = x\varphi_{i\tau}(x)$. Recall that for $f \in L^2(\mathbb{R})$ the Weil-Brezin transform is given by

$$Vf(x,y,t) = e^{2\pi i t} \sum_{n=-\infty}^{\infty} f(x+n)e^{2\pi i n y}.$$

Definition 2.13. *The theta transform is defined on* $L^2(M)$ *by*

$$\Theta(F,\tau) = (V\varphi_{i\tau}, F) = \int_M V\varphi_{i\tau}(g)\bar{F}(g)dg.$$

We also define $\Theta^*(F,\tau) = (V\psi_{i\tau}, F)$.

Since V is a unitary operator we get the formulas

$$\Theta(V\bar{f},\tau) = \int_{-\infty}^{\infty} f(x)e^{\pi i\tau x^2}dx$$

and

$$\Theta^*(V\bar{f},\tau) = \int_{-\infty}^{\infty} xf(x)e^{\pi i\tau x^2}dx$$

Note that $\Theta(F,\tau)$ and $\Theta^*(F,\tau)$ are functions defined on the upper half-plane $\mathbb{R}^2_+ = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$.

Theorem 2.14. *For* $F \in L^2(M)$ *both* $\Theta(F,\tau)$ *and* $\Theta^*(F,\tau)$ *are holomorphic in the upper half-plane.*

Proof. It is clear that $\Theta(F,\tau)$ is holomorphic in the upper half-plane when $F \in C^\infty(M)$. If $F_n \in C^\infty(M)$ converges to $F \in L^2(M)$ then $\Theta(F_n,\tau)$ converges to $\Theta(F,\tau)$ uniformly over compact subsets of the upper half-plane. This follows from the fact that $V(\varphi_{i\tau})$ is bounded when τ is restricted to compact subsets. This shows that $\Theta(F,\tau)$ is holomorphic. The proof for $\Theta^*(F,\tau)$ is the same. □

We can decompose $\mathcal{H}_1$ as $\mathcal{H}_1^o \oplus \mathcal{H}_1^e$ where $\mathcal{H}_1^o$ (resp. $\mathcal{H}_1^e$) is the image under V of all odd (even) functions. Note that $V\varphi_{i\tau} \in \mathcal{H}_1^e$ and $V\psi_{i\tau} \in \mathcal{H}_1^e$. Moreover, $\Theta(F,\tau) = 0$ if $F \in \mathcal{H}_1^o$ and $\Theta^*(F,\tau) = 0$ if $F \in \mathcal{H}_1^e$. We can now prove the following uniqueness theorem for the theta transform.

Theorem 2.15. *For* $F \in \mathcal{H}_1$, $F = 0$ *if and only if* $\Theta(F,\tau) = \Theta^*(F,\tau) = 0$ *for all* $\tau \in \mathbb{R}^2_+$. *Consequently, the set of all functions* $\{V\varphi_{i\tau}, V\psi_{i\tau}, \tau \in \mathbb{R}^2_+\}$ *is dense in* $\mathcal{H}_1$.

Proof. If $F = G + H$ where $G \in \mathcal{H}_1^e$ and $H \in \mathcal{H}_1^o$ then $\Theta(F,\tau) = \Theta(G,\tau)$. Therefore, $\Theta(F,\tau) = 0$ for all τ gives, by taking $\tau = t + i$,

$$\int_0^{\infty} g(x)e^{-\pi x^2}e^{\pi itx^2}dx = 0$$

where $G = V(\bar{g})$. By making a change of variables we get

$$\int_0^\infty g(s^{\frac{1}{2}})s^{-\frac{1}{2}}e^{-\pi s}e^{ist}ds = 0.$$

As the integrand belongs to $L^1(\mathbb{R})$ by the uniqueness theorem for the Fourier transform we get $g = 0$. Similarly, the other condition $\Theta^*(F,\tau) = 0$ gives $h = 0$. Hence the theorem. □

Corollary 2.16. *The Hermite functions $\{\varphi_k : k \in \mathbb{N}\}$ form an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. It is enought to show that the set of all functions $\{t^n e^{-\pi t^2} : n \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$. Suppose f is orthogonal to all these functions. Let $F = V(\bar{f})$ and consider $\theta(\tau) = \Theta(F,\tau)$. Evaluating the derivatives of θ at $\tau = i$ we get

$$\theta^{(n)}(i) = \int_{-\infty}^{\infty} f(t)t^{2n}e^{-\pi t^2}dt = 0.$$

As θ is holomorphic we get $\Theta(F,\tau) = 0$. As before, if $F = G + H, G \in \mathcal{H}_1^e, H \in \mathcal{H}_1^0$ we have $\Theta(G,\tau) = \Theta(F,\tau) = 0$ and $\Theta^*(G,\tau) = 0$. Hence $G = 0$. This means that f is odd. Working with $\Theta^* V(\bar{f})$ we can also conclude that f is even. Hence $f = 0$ proving the corollary. □

We now use properties of the theta transform to prove a result on Fourier tranform pairs due to Hardy. This result will be used to construct some more examples of invariant and ultravariant subspaces.

Theorem 2.17. *Suppose $f \in L^2(\mathbb{R})$ satisfies the growth conditions*

$$|f(x)| \leq Ce^{-\pi t x^2}, \ |\mathcal{F}f(y)| \leq Ce^{-\frac{\pi}{t}y^2}$$

for some $t > 0$. Then $f(x) = Ce^{-\pi t x^2}$.

Proof. By dilating by t we can assume that $t = 1$. Recalling the definitions of $\varphi_{i\tau}$ and $\psi_{i\tau}$ we can easily calculate that

$$\Theta(V\varphi_{-1},\tau) = (1 - i\tau)^{-\frac{1}{2}}, \ \Theta^*(V\psi_{-1},\tau) = (1 - i\tau)^{-\frac{3}{2}}.$$

Given f as in the theorem we can write it as $f = g + h$, where $g(x) = \frac{1}{2}(f(x) + f(-x))$ and $h(x) = \frac{1}{2}(f(x) - f(-x))$. Observe that both g

and h satisfy the same growth conditions as f. We therefore, prove the theorem for even and odd functions separately.

If f is even, consider the decomposition $f = g + h$ where $g = \frac{1}{2}(f + \mathcal{F}f)$ and $h = \frac{1}{2}(f - \mathcal{F}f)$. Then $\mathcal{F}g = g, \mathcal{F}h = -h$ and both satisfy the conditions of the theorem. If f is odd the decompostion $g = \frac{1}{2}(f + i\mathcal{F}f)$ and $h = \frac{1}{2}(f - i\mathcal{F}f)$ gives $f = g + h$ with $\mathcal{F}g = -ig$ and $\mathcal{F}h = ih$. This shows that we can assume without loss of generality f is an eigenfunction of $\mathcal{F}$. We start with the even case, $\mathcal{F}f = cf$ where $c = 1$ or -1.

We consider the function $\alpha(\tau) = \Theta(V(\bar{f}), \tau)$ which is given by the integral

$$\alpha(\tau) = \int_{-\infty}^{\infty} f(x) e^{\pi i \tau x^2} dx.$$

The growth condition on f shows that $\alpha(\tau)$ is holomorphic in $\Im(\tau) > -1$. Since

$$\Theta(V(\bar{f}), \tau) = (V\varphi_{i\tau}, V\bar{f}) = (\varphi_{i\tau}, \bar{f})$$

using the result $(f, g) = (\mathcal{F}f, \mathcal{F}g)$ we get

$$\Theta(V(\bar{f}), \tau) = (-i\tau)^{-\frac{1}{2}} (\varphi_{\frac{-i}{\tau}}, \bar{f}) = c(-i\tau)^{-\frac{1}{2}} \alpha(\frac{-1}{\tau}).$$

In the above calculation we have used the facts that $\mathcal{F}\varphi_{i\tau} = (-i\tau)^{-\frac{1}{2}} \varphi_{\frac{-i}{\tau}}$ and $\mathcal{F}\bar{f} = c\bar{f}$. Therefore, α satisfies $\alpha(\tau) = c(-i\tau)^{-\frac{1}{2}} \alpha(\frac{-1}{\tau})$ provided both $\Im(\tau) > -1$ and $\Im(\frac{-1}{\tau}) > -1$.

Define a new function $\beta(\tau) = (1 - i\tau)^{\frac{1}{2}} \alpha(\tau)$. If we can show that $\beta(\tau)$ is a constant which means that $\alpha(\tau) = C\Theta(V\varphi_{-1}, \tau)$ then by the uniqueness theorem for the theta transform we get $f = \varphi_{-1}$. This will take care of the even case.

An easy calculation shows that the function β satisfies $\beta(\tau) = c\beta(\frac{-1}{\tau})$ whenever both $\Im(\tau) > -1$ and $\Im(\frac{-1}{\tau}) > -1$. Define $\gamma(\tau)$ by the setting it equal to $\beta(\tau)$ when $\Im(\tau) > -1$ and $c\beta(\frac{-1}{\tau})$ when $\Im(\frac{-1}{\tau}) > -1$. If $\tau = a + ib, b \leq -1$, then $-b = |b| < b^2 + a^2$ so that $\Im(\frac{-1}{\tau}) = \frac{b}{(a^2+b^2)} > -1$ as long as $a \neq 0$. This means that $\gamma(\tau)$ can be extended to the entire complex plane except possibly $\tau = -i$. With $\tau = a + ib$ we have the estimate $|\alpha(\tau)| \leq C(1 + b)^{-\frac{1}{2}}, b > -1$ which follows from the integral defining α and the hypothesis on f. When $a^2+b^2 \leq 1$ writing $b = -1+\delta, \delta > 0$ we have $a^2 \leq 1-b^2 = \delta(2-\delta)$. This gives the estimate

$$|(1-i\tau)\gamma(\tau)| \leq C(1+b)(1+\frac{|a|}{1+b})^{\frac{3}{2}}$$

$$\leq C\delta(1+2\delta^{-\frac{1}{2}})^{\frac{3}{2}} \leq C\delta^{\frac{1}{4}}(2+\delta^{\frac{1}{2}})^{\frac{3}{2}}.$$

This together with the property $\gamma(\tau) = c\gamma(\frac{-1}{\tau})$ shows that $(1-i\tau)\gamma(\tau)$ tends to zero as τ goes to $-i$. Hence γ is entire. It is also bounded (since $\gamma(\tau) = c\gamma(\frac{-1}{\tau})$) and hence γ reduces to a constant. This proves that β is a constant.

When f is an odd eigenfunction of the Fourier transform we work with $(1-i\tau)^{\frac{3}{2}}\Theta^*(V\bar{f},\tau)$ and show that $f(x) = Cxe^{-\pi x^2}$. But now the constant C has to be zero if the growth condition on f is satisfied. Thus the odd component of f is zero proving the theorem.

□

2.5. **Fourier transforms of L^p functions.** So far we have considered Fourier transforms of functions which are either integrable or square integralble. It is therefore natural to ask if we can define Fourier transforms of functions coming from other L^p spaces. There is a very simple way of defining Fourier transforms of L^p functions when $1 < p < 2$. For $p > 2$ we cannot define Fourier transform on L^p without some knowledge of distributions. So we concentrate mainly on $L^p(\mathbb{R}), 1 < p < 2$.

We start withe the observation that any $f \in L^p(\mathbb{R}), 1 < p < 2$ can be decomposed as $f = g + h$ where $g \in L^1(\mathbb{R})$ and $h \in L^2(\mathbb{R})$. Indeed, we can simply define $g(x) = f(x)$ when $|f(x)| > 1$, $g(x) = 0$ otherwise and let $h = f - g$. Then it is easily verified that $g \in L^1(\mathbb{R})$ and $h \in L^2(\mathbb{R})$. The decomposition is clearly not unique. Nevertheless, it allows us to define the Fourier transform of f by $\hat{f} = \hat{g} + \mathcal{F}h$. We need to check that this definition is independent of the decomposition. If we have $f = g_i + h_i, g_i \in L^1(\mathbb{R}), h_i \in L^2(\mathbb{R}), i = 1, 2$ then $g_1 - g_2 = h_2 - h_1$ so that $\hat{g_1} - \hat{g_2} = \mathcal{F}h_2 - \mathcal{F}h_1$. Hence $\hat{g_1} + \mathcal{F}h_1 = \hat{g_2} + \mathcal{F}h_2$ proving that $\hat{f}$ is well defined.

From the definition it follows that $\hat{f}$ for $f \in L^p(\mathbb{R}), 1 < p < 2$ is a sum of an L^∞ function and an L^2 function. We also know that the Fourier transform takes L^1 continuously into L^∞ and L^2 onto L^2. An application of an interpolation theorem of Riesz and Thorin, which we do not prove, shows that the Fourier transform takes L^p continuously into L^q where $1 < p < 2, p + q = pq$. The resulting inequality $\|\hat{f}\|_q \leq \|f\|_p$ is known as the Hausdorff-Young inequality for the Fourier transform.

In this section we are mainly interested in the problem of reconstructing the function f from its Fourier transform. On L^2 the inversion formula $f = \mathcal{F}^*\mathcal{F}f$ does this. When both f and $\hat{f}$ are integrable we also have the inversion formula

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

which holds for almost every x. For $f \in L^p(\mathbb{R}), 1 \leq p \leq 2$ we define, for every $R > 0$,

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

which is the analogue of the partial sums $'S_n f'$ for the Fourier series. Note that the integral converges and $S_R f$ well defined- this is clear since

$$S_R f(x) = \int_{\mathbb{R}} \chi_{(-R,R)}(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

, $\hat{f} \in L^q(\mathbb{R})$ and $\chi_{(-R,R)} \in L^p(\mathbb{R})$. We are interested in knowing if $S_R f$ converges f in the norm as R tends to infinity. As in the case of Fourier series the L^2 case is easy to settle.

Theorem 2.18. *For every $f \in L^2(\mathbb{R})$ the partial sums $S_R f$ converge to f in the L^2 morm as R tends to infinity.*

Proof. We follow the standard strategy of proving uniform boundedness of S_R on $L^2(\mathbb{R})$ and convergence of $S_R f$ to f on a dense subspace of $L^2(\mathbb{R})$. The uniform boundedness follows from the Plancherel theorem:

$$\|S_R f\|_2^2 = \int_{\mathbb{R}} \chi_{(-R,R)}(\xi)|\hat{f}(\xi)|^2 d\xi \leq \|f\|_2^2.$$

Let $C_c(\mathbb{R})$ be the space of all continuous functions with compact support which is dense in L^2. Define

$$W = \{g \in L^2(\mathbb{R}) : \mathcal{F}g \in C_c(\mathbb{R})\}.$$

The density of $C_c(\mathbb{R})$ in L^2 and Plancherel theorem shows that W is also dense in L^2. Moreover, for $f \in W$ with $\hat{f}$ supported in $(-a, a)$ we have

$$S_R f(x) = \int_{-a}^{a} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = f(x)$$

as soon as $R > a$. Thus, $S_R f(x)$ converges to $f(x)$ almost everywhere. Further,

$$\|S_R f - S_{R'} f\|_2^2 = \int_{R \leq |\xi| \leq R'} |\hat{f}(\xi)|^2 d\xi$$

shows that $S_R f$ is Cauchy in L^2 and hence converges to some L^2 function. As it already converges to f almost everywhere, the L^2 limit has to be f itself. Thus $S_R f$ converges to f in L^2 for all $f \in W$. This completes the proof of the theorem. □

We now turn our attention to $S_R f$ for functions from $L^p(\mathbb{R})$. We define the continuous analogue of the Dirichlet kernel by

$$s_R(x) = \int_{\mathbb{R}} \chi_{(-R,R)}(\xi) e^{-2\pi i x \xi} d\xi.$$

A simple calculation shows that

$$s_R(x) = R\left(\frac{\sin(2\pi R x)}{\pi R x}\right).$$

Therefore, $s_R \in L^q(\mathbb{R})$ for all $q \geq 2$ and hence the convolution

$$f * s_R(x) = \int_{\mathbb{R}} f(x - y) s_R(y) dy$$

makes sense for all $f \in L^p(\mathbb{R}), 1 \leq p \leq 2$. We claim that $S_R f = f * s_R$ for all $f \in L^1 \cap L^p(\mathbb{R})$. To prove this, consider the two operators defined on $L^1 \cap Ł^2(\mathbb{R})$ by $Tg = \mathcal{F}(f * g)$ and $Sg = \mathcal{F}(f)\mathcal{F}(g)$. Clearly, both extend to L^2 as bounded linear operators and by direct calculation $Tg = Sg$ for all $g \in L^1 \cap Ł^2(\mathbb{R})$. Therefor, $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ for all $g \in L^2$. this proves our claim.

As in the case of Fourier series we show that S_R are not uniformly bounded on $L^1(\mathbb{R})$. To prove this we first establish an analogue of Fejer's theorem for the Fourier transform. We define, for all $f \in W$ (say)

$$\sigma_R f(x) = \int_{\mathbb{R}} \lambda_R(\xi)\hat{f}(\xi)e^{2\pi i x\xi} d\xi$$

where

$$\lambda_R(\xi) = (2R)^{-1} \int_{\mathbb{R}} \chi_{(-R,R)}(\xi - \eta)\chi_{(-R,R)}(\eta)d\eta.$$

It is easy to see that $\lambda_R(\xi)$ is a continuos function supported in $(-2R, 2R)$ and $\lambda_R(0) = 1$. Naturally, we expect $\sigma_R f$ to converge to f. Since $\hat{\lambda_R}(x) = (2R)^{-1}(s_R(x))^2$ it follows that $\hat{\lambda_R}$ is integrable with integral one. Consequently, $\sigma_R f = f * \hat{\lambda_R}$. We have the following analogue of Fejer's theorem.

Theorem 2.19. *For every $f \in L^p(\mathbb{R}), 1 \leq p < \infty, \sigma_R f$ converges to f in the norm.*

Proof. Since $\hat{\lambda_R}$ is integrable, we have $\|\sigma_R f\|_p \leq \|f\|_p$. Therefore, it is enough to show that $\sigma_R f$ converges to f in the norm for all f in $C_c(\mathbb{R})$ which is dense in every $L^p(\mathbb{R}), 1 \leq p < \infty$. As integral of $\hat{\lambda_R}$ is one we have

$$\sigma_R f(x) = \int_{\mathbb{R}} (f(x - y) - f(x))\hat{\lambda_R}(y)dy.$$

By Minkowski's integral inequality, it follows that

$$\|\sigma_R f\|_p \leq \int_{\mathbb{R}} \|f(\cdot - y) - f\|_p \hat{\lambda_R}(y)dy.$$

When $|y| > 1$ we have the estimate $\hat{\lambda_R}(y) \leq CR^{-1}|y|^{-2}$ and consequently

$$\int_{|y|>1} \|f(\cdot - y) - f\|_p \hat{\lambda_R}(y)dy \leq C\|f\|_p R^{-1}$$

which goes to zero as R approaches infinity. On the other hand the rest of the integral is bounded by

$$\int_{|y|\leq R} \|f(\cdot - y/R) - f\|_p \hat{\lambda_1}(y)dy.$$

Since f is compactly supported we can choose R large enough so that $\|f(\cdot - y/R) - f\|_p$ is uniformly small for all y in the support of f. This completes the proof. □

We can now show that S_R are not uniformly bounded on $L^1(\mathbb{R})$. First we remark that s_R is not integrable. To see this, suppose $s_R \in L^1(\mathbb{R})$. then by Fourier inversion we should have

$$\chi_{(-R,R)}(x) = \int_{\mathbb{R}} s_R(\xi) e^{2\pi i x\xi} d\xi$$

for almost every x. But $\chi_{(-R,R)}(x)$ cannot be equal to equal to any continuous function almost everywhere as it has jump discontinuities at R and $-R$.

Theorem 2.20. *The partial sum operators S_R are not uniformly bounded on $L^1(\mathbb{R})$.*

Proof. Suppose $\|S_R f\|_1 \leq C\|f\|_1$ for all $f \in L^1(\mathbb{R})$ with C independent of R. Taking $f = \hat{\lambda}_R$ we get $\|\hat{\lambda}_R * s_1\|_1 \leq C$. As R tends to infinity, $\hat{\lambda}_R * s_1$ converges to s_1 in L^2 and hence almost everywhere along a subsequence. In view of Fatou's lemma this leads to the conclusion that

$$\int_{\mathbb{R}} |s_1(x)| dx \leq C \liminf_{R\to\infty} \int_{\mathbb{R}} |\hat{\lambda}_R * s_1(x) dx \leq C$$

which is a contradiction since s_1 is not integrable.

□

It turns out that the partial sum operators S_R defined on $L^1 \cap L^p(\mathbb{R})$ extends to the whole of $L^p(\mathbb{R})$ as a bounded linear operator for all values of $p, 1 < p < \infty$. The proof is not easy- one way to prove this result is to use the corresponding result on the partial sum operators S_n for the Fourier series. We will return to this problem later but now we look at a related problem.

The operators S_R and σ_R both have one thing in common: both are defined by multiplying $\hat{f}$ by a bounded function and then inverting the

Fourier transform. More generally we can define operators of the form

$$T_m f(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

where m is a bounded function. This kind of operators are called multiplier transforms and the function m is called a Fourier multiplier. Note that $T_m f$ makes sense for all $f \in W \cap L^1(\mathbb{R})$ where f is the subspace defined earlier in this section. As m is bounded it is clear that T_m extends to $L^2(R)$ as a bounded operator and $\|T_m f\|_2 \leq \|m\|_\infty \|f\|_2$. But the operator may not extend to other L^p spaces as a bounded operator. We already have the example of S_R corresponding to the multiplier $m = \chi_{(-R,R)}$.

An important class of multiplier studied in the literature is the so called Bochner-Riesz means S_R^δ corresponding to the multiplier

$$m_R^\delta(\xi) = (1 - |\xi|^2/R^2)_+^\delta = (1 - |\xi|^2/R^2)^\delta \chi_{(-R,R)}(\xi)$$

where $\delta \geq 0$. In other words,

$$S_R^\delta f(x) = \int_{\mathbb{R}} (1 - |\xi|^2/R^2)_+^\delta \hat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

We note that $S_R^0 = S_R$ and the multiplier becomes smoother and smoother as δ increases. The main result we want to prove is the following.

Theorem 2.21. *Let $\delta > 0$. Then S_R^δ, initially defined on $L^1 \cap L^p(\mathbb{R})$ extend to $L^p(\mathbb{R})$ as uniformly bounded operators for all $1 \leq p < \infty$. As R tends to infinity, $S_R^\delta f$ converge to f in the norm.*

Proof. We show that $S_R^\delta f = f * s_R^\delta$ with an explicit kernel. By studying the asymptotics we show that the kernel is uniformly integrable as soon as $\delta > 0$. The theorem then follows from Young's inequality for convolution. □

As in the case of partial sums and Fejer means the kernel is simply defined by

$$s_R^\delta(x) = \int_{-R}^{R} (1 - |\xi|^2/R^2)^\delta e^{2\pi i x\xi}d\xi.$$

Observe that $s_R^\delta(x) = Rs_1^\delta(Rx)$ and therefore our claim about the uniform integrability of the kernel follows from the next lemma.

Lemma 2.22. *For any $\delta \geq 0$ we have*

$$|\int_{-1}^{1}(1-s^2)^\delta e^{its}ds| \leq C(1+t^2)^{-\frac{1}{2}(\delta+1)}.$$

Proof. The integral is clearly bounded and hence we only need to show that it is bounded by $C|t|^{-\delta-1}$ for $|t| > 1$. As the integral is an even function of t we can assume that t is positive. Consider the domain in $\mathbb{C}$ obtained by removing the intervals $(-\infty, -1)$ and $(1, \infty)$. Let $F(z) = (1-z^2)^\delta$ be the holomorphic function in that region which is real and positive on $(-1, 1)$. Integrating F along the boundary of the rectangle with corners at $(-1, 0), (1, 0), (1, a)$ and $(-1, a)$ with $a > 0$ and using Cauchy's theorem we get

$$\int_{-1}^{1}(1-x^2)^\delta e^{itx}dx - \int_{-1}^{1}(1-(x+ia)^2)^\delta e^{it(x+ia)}dx$$

$$= -i\int_0^a(1-(1+iy)^2)^\delta e^{it(1+iy)}dy + i\int_0^a(1-(-1+iy)^2)^\delta e^{it(-1+iy)}dy.$$

As a tends to infinity the second integral on the left hand side goes to zero leaving us with

$$\int_{-1}^{1}(1-x^2)^\delta e^{itx}dx = -ie^{it}\int_0^\infty(y^2-2iy)^\delta e^{-ty}dy + ie^{-it}\int_0^\infty(y^2+2iy)^\delta e^{-ty}dy.$$

The integrals on the right hand side are clearly bounded by

$$C\int_0^\infty(y^\delta + y^{2\delta})e^{-ty}dy \leq C(t^{-\delta-1}+t^{-2\delta-1})$$

which proves the lemma.

□

For any $\delta > -1/2$ the Bessel function J_δ is defined by

$$J_\delta(t) = \frac{(t/2)^\delta}{\Gamma(\delta+1/2)\Gamma(1/2)}\int_{-1}^{1}(1-s^2)^{\delta-1/2}e^{its}ds.$$

In terms of the Bessel function we have

$$s_R^\delta(x) = \Gamma(\delta+1/2)\Gamma(1/2)R(\pi R|x|)^{-\delta-1/2}J_{\delta+1/2}(2\pi R|x|).$$

The above lemma shows that $J_\delta(t)$ decays like $t^{-1/2}$ as t tends to infinity for all $\delta \geq 1/2$. It can be shown that the same is true for all $\delta > -1/2$.

3. Fourier transform on $\mathbb{R}^n$

In the previous section we defined the Fourier transform on functions defined on $\mathbb{R}$. We can easily extend the definition to functions on $\mathbb{R}^n$. Instead of $\mathbb{H}^1$ we consider the $(2n+1)$ dimensional group $\mathbb{H}^n$ which is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + x \cdot y').$$

Let $\Gamma = \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$. Then Γ is a subgroup of $\mathbb{H}^n$ so that we can form the quotient $M = \Gamma/\mathbb{H}^n$ consisting of all right cosets of Γ. Functions on M are naturally identified with left $\Gamma-$invariant functions on $\mathbb{H}^n$. As the Lebesgue measure $dxdydt$ is left $\Gamma-$ invariant we can form $L^2(M)$ using the Lebesgue measure restricted to M. As a set we can identify M with $[0,1)^{2n+1}$ and we just think of $L^2(M)$ as $L^2([0,1)^{2n+1})$.

As before Fourier expansion in the last variable allows us to decompose $L^2(M)$ into a direct sum of orthogonal subspaces. Simply define $\mathcal{H}_k$ to be the set of all $f \in L^2(M)$ which satisfy the condition

$$f(x, y, t + s) = e^{2\pi iks} f(x, y, t).$$

Then $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_j$ whenever $k \neq j$ and any $f \in L^2(M)$ has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k, \;\; f_k \in \mathcal{H}_k.$$

We are mainly interested in $\mathcal{H}_1$ which is a Hilbert space in its own right.

Define J as in the one dimesional case and let $V : L^2(\mathbb{R}^n) \to \mathcal{H}_1$ be defined by

$$Vf(x, y, t) = e^{2\pi it} \sum_{m \in \mathbb{Z}^n} f(x + m) e^{2\pi i m \cdot y}.$$

Then V is unitary and we simply define $\mathcal{F} = V^*JV$. It is then clear that $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}^n)$. All the results proved for functions on $\mathbb{R}$ remain true now. We record the following two theorems.

Theorem 3.1. *The Fourier transform $\mathcal{F}$ satisfies: (i) $\mathcal{F}^4 f = f$, for every $f \in L^2(\mathbb{R}^n)$ (ii) $\mathcal{F}^2 f(x) = f(-x)$ for almost every $x \in \mathbb{R}^n$ and (iii) $\|\mathcal{F}f\|_2 = \|f\|_2$.*

Theorem 3.2. *For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ the Fourier transform is given by*

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx.$$

If we further assume that $\mathcal{F}f \in L^1(\mathbb{R}^n)$ then for almost every x we have

$$f(x) = \int_{\mathbb{R}^n} \mathcal{F}f(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

The eigenfunctions of $\mathcal{F}$ are obtained by taking tensor products of the one dimensional Hermite functions. For each $\alpha \in \mathbb{N}^n$ we define

$$\Phi_\alpha(x) = \varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)...\varphi_{\alpha_1}(x_n).$$

Then it follows that $\mathcal{F}\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$ where $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Thus every $f \in L^2(\mathbb{R}^n)$ has an expansion

$$f = \sum_{\alpha\in\mathbb{N}^n} (f, \Phi_\alpha)\Phi_\alpha$$

the series being convergent in the L^2 sense. Defining $P_k f = \sum_{|\alpha|=k}(f, \Phi_\alpha)\Phi_\alpha$ we can write the above in the compact form $f = \sum_{k=0}^\infty P_k f$. Note that $\mathcal{F}(P_k f) = (-i)^k P_k f$. The Hermite functions Φ_α are eigenfunctions of the Hermite operator $H = -\Delta + 4\pi^2|x|^2$ where Δ is the Laplacian. More precisely, $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$ where $|\alpha| = \sum_{j=1}^n \alpha_j$. Thus $P_k f$ is just the orthogonal projection of f on the k- th eigenspace spanned by $\{\Phi_\alpha : |\alpha| = k\}$.

We now look at some subspaces of $L^2(\mathbb{R}^n)$ that are left invariant under the action of Fourier transform. We make a formal definition:

Definition 3.3. *A subspace W of $L^2(\mathbb{R}^n)$ is said to be invariant under $\mathcal{F}$ if $\mathcal{F}f \in W$ whenever $f \in L^2(\mathbb{R}^n)$.*

Some examples of such subspaces include the Hermite-Sobolev spaces, the Schwartz space, the Hermite-Bergman spaces and eigenspaces of the Fourier transform.

3.1. **The Schwartz space.** Our first example of an invariant subspace is provided by the spaces of Schwartz functions. As $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ it follows that $f \in L^2(\mathbb{R}^n)$ if and only if $\sum_{\alpha\in\mathbb{N}^n} |(f, \Phi_\alpha)|^2 < \infty$. Since $(\mathcal{F}f, \Phi_\alpha) = (-i)^{|\alpha|}(f, \Phi_\alpha)$ any subspace of $L^2(\mathbb{R}^n)$ defined in terms of the behaviour of $|(f, \Phi_\alpha)|$ will be invariant under the Fourier transform. We can define a whole family of invariant subspaces. Indeed, for each $s > 0$ define $W^s_H(\mathbb{R}^n)$ to be the subspace of $L^2(\mathbb{R}^n)$ consisting of functions f for which

$$\|f\|^2_{2,s} = \sum_{\alpha\in\mathbb{N}^n} (2|\alpha| + n)^s |(f, \Phi_\alpha)|^2 < \infty.$$

We let $\mathcal{S}(\mathbb{R}^n) = \cap_{s>0} W^s_H(\mathbb{R}^n)$. This is called the Schwartz space, members of which are called Schwartz functions.

Theorem 3.4. *The Schwartz space has the following properties. (i) $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^2(\mathbb{R}^n)$; (ii) $\mathcal{S}(\mathbb{R}^n)$ is invariant under $\mathcal{F}$ and (iii) $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is one to one and onto.*

The density follows from the fact that finite linear combinations of Hermite functions form a subspace of $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$. The invariance follows from that of each of $W^s_H(\mathbb{R}^n)$. As $\mathcal{F}(\mathcal{F}^*f) = f$ the surjectivity is proved.

According to our definition a function $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if $|(f, \Phi_\alpha)| \leq C_m(2|\alpha| + n)^{-m}$ for all $m \in \mathbb{N}$. It is desirable to have an equivalent definition which does not involve the Hermite coefficients. The Hermite functions Φ_α are known to be uniformly bounded, i.e., $\|\Phi_\alpha\|_\infty \leq C$ for all α. In view of this we see that the series

$$f(x) = \sum_{\alpha\in\mathbb{N}^n} (f, \Phi_\alpha)\Phi_\alpha(x)$$

converges uniformly whenever $f \in W_H^s(\mathbb{R}^n)$ with $s > n$. Indeed, writing

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} (f, \Phi_\alpha)(2|\alpha| + n)^{\frac{s}{2}} (2|\alpha| + n)^{-\frac{s}{2}} \Phi_\alpha(x),$$

applying Cauchy-Schwarz inequality and using the fact that

$$\sum_{|\alpha|=k} 1 = \frac{(k+n-1)!}{k!(n-1)!} = O(k^{n-1})$$

we get

$$|f(x)|^2 \leq \|f\|_s^2 \sum_{k=0}^{\infty} (2k+n)^{-s+n-1} \leq C\|f\|_s^2.$$

More generally we have the following theorem known as Sobolev embedding theorem.

Theorem 3.5. *Suppose $f \in W_H^s(\mathbb{R}^n)$ with $s > n+m$. Then $\|\partial^\beta f\|_\infty \leq C_m \|f\|_s$ for all $|\beta| \leq m$.*

In order prove this theorem we need the following properties of the Hermite functions. Let $A_j = -\frac{\partial}{\partial x_j} + 2\pi x_j$ and $B_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, ..., n$. Let e_j be the coordinate vectors in $\mathbb{R}^n$ with 1 in the j-th place. It can be verified that $A_j B_j + B_j A_j = 2H_j$ and $A_j B_j - B_j A_j = -4\pi I$ where H_j is the Hermite operator in x_j. Thus, $2H = \sum_{j=1}^n (A_j B_j + B_j A_j)$.

Proposition 3.6. *For any $\alpha \in \mathbb{N}^n$ and $j = 1, 2, ..., n$, $A_j \Phi_\alpha = 2\pi^{\frac{1}{2}} (\alpha_j + 1)^{\frac{1}{2}} \Phi_{\alpha+e_j}$, $B_j \Phi_\alpha = 2\pi^{\frac{1}{2}} (\alpha_j)^{\frac{1}{2}} \Phi_{\alpha-e_j}$.*

Proof. It is enough to prove the proposition when $n = 1$. Let us write $A = A_1$ and $B = B_1$. From the relation $AB - BA = -4\pi I$ we obtain $A^2 B - BA^2 = -8\pi A$. Another simple calculation shows that $HA = A(H + 4\pi I)$ which leads to the conclusion that $A\varphi_k$ is an eigen function of H with eigenvalue $2\pi(2k+3)$ and hence $A\varphi_k = c_k \varphi_{k+1}$ for a constant c_k. Similarly, we can show that $B\varphi_k = d_k \varphi_{k-1}$. Since $BA = H + 2\pi I$ we get $c_k d_{k+1} = 2\pi(2k+2)$. On the other hand, $2x\varphi_k(x) = (A+B)\varphi_k(x) = c_k \varphi_{k+1}(x) + d_k \varphi_{k-1}(x)$ so that

$$c_k = \int_{\mathbb{R}} 2x \varphi_k(x) \varphi_{k+1}(x) dx = d_{k+1}.$$

Hence we get $c_k^2 = 2\pi(2k+2)$ and $d_k^2 = 2\pi(2k)$ proving the proposition. □

The Sobolev embedding theorem follows from the above properties of the Hermite functions. Since $B_j - A_j = 2\frac{\partial}{\partial x_j}$ it follows that

$$\partial^\beta \Phi_\alpha = 2^{-|\beta|}(B-A)^\beta \Phi_\alpha$$

where $B - A = (B_1 - A_1, ..., B_n - A_n)$. From the proposition it follows that

$$|\partial^\beta \Phi_\alpha(x)| \leq C(2|\alpha| + n)^{\frac{1}{2}|\beta|}.$$

This estimate, along with Cauchy-Schwarz, proves the theorem.

The Sobolev embedding theorem leads to the following useful characterisation of the Schwartz space.

Theorem 3.7. *A function $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^n)$ for all α and β.*

Proof. The condition $x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^n)$ for all α and β is clearly equivalent to $x^\alpha \partial^\beta f \in L^2(\mathbb{R}^n)$ for all α and β. This implies that $H^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and consequently $|(f, \Phi_\alpha)| \leq C_m(2|\alpha| + n)^{-m}$ for all α. Thus, $f \in W_H^m(\mathbb{R}^n)$ for all m and hence $f \in \mathcal{S}(\mathbb{R}^n)$. To prove the converse, suppose $f \in \mathcal{S}(\mathbb{R}^n)$ so that $f \in W_H^m(\mathbb{R}^n)$ for all $m \in \mathbb{N}$. We need to consider

$$x^\alpha \partial^\beta f(x) = \sum_{\mu \in \mathbb{N}^n} (f, \Phi_\mu) x^\alpha \partial^\beta \Phi_\mu(x).$$

Using the relations $2x_j = (A_j + B_j)$ and $2\partial_j = (B_j - A_j)$ we can express $x^\alpha \partial^\beta \Phi_\mu(x)$ in terms of $(B-A)^\gamma \Phi_\mu$. Using the proposition and Cauchy-Schwarz inequality we can easily check that $x^\alpha \partial^\beta f(x)$ is bounded. □

The Schwartz space can be made into a locally convex topological vector space such that $\mathcal{S}(\mathbb{R}^n)$ is continuously embedded in $W_H^s(\mathbb{R}^n)$ for every $s > 0$. The dual of $\mathcal{S}(\mathbb{R}^n)$ denoted by $\mathcal{S}'(\mathbb{R}^n)$ is called the space of tempered distributions. It can be shown that $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only if

$$\sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{-s} |(f, \Phi_\alpha)|^2 < \infty$$

for some $s > 0$. The Fourier transform has a natural extension to $\mathcal{S}'(\mathbb{R}^n)$ given by $(\mathcal{F}u, f) = (u, \mathcal{F}f)$ where the brackets here stand for the action of a tempered distribution on a Schwartz function.

3.2. **Weighted Bergman spaces.** From the definition it follows that $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if for every $s > 0$, there are constants C_s such that

$$|(f, \Phi_\alpha)| \leq C_s(2|\alpha| + n)^{-s}, \ \alpha \in \mathbb{N}^n.$$

It is natural to consider conditions of the form

$$|(f, \Phi_\alpha)| \leq Ce^{-(2|\alpha|+n)s}, \ \alpha \in \mathbb{N}^n$$

for some $s > 0$. This will lead to another family of invariant subspaces which can be identified with certain Hilbert spaces of entire functions. For each $t > 0$ let us consider the weight function

$$U_t(x, y) = 2^n(\sinh(4t))^{-\frac{n}{2}} e^{2\pi(\tanh(2t)|x|^2 - \coth(2t)|y|^2)}.$$

We define $\mathcal{H}_t(\mathbb{C}^n)$ to be the subspace of entire functions satisfying

$$\|F\|^2_{\mathcal{H}_t} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x + iy)|^2 U_t(x, y) dx dy < \infty.$$

These are examples of weighted Bergman spaces. We call them Hermite-Bergman spaces for reasons which will become clear soon. We let $\mathcal{H}_t(\mathbb{R}^n)$ to stand for the space of all restrictions of $F \in \mathcal{H}_t(\mathbb{C}^n)$ so that we can think of $\mathcal{H}_t(\mathbb{R}^n)$ as a subspace of $L^2(\mathbb{R}^n)$.

Theorem 3.8. *For each $t > 0$ the space $\mathcal{H}_t(\mathbb{R}^n)$ is invariant under the Fourier transform.*

Since $\Phi_\alpha(x) = H_\alpha(x)e^{-\pi|x|^2}$ where H_α is a polynomial we can extend Φ_α to $\mathbb{C}^n$ as an entire function simply by setting $\Phi_\alpha(z) = H_\alpha(z)e^{-\pi z^2}$ where $z^2 = \sum_{j=1}^n z_j^2, z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$. The reader can verify by direct calulation that $\Phi_\alpha \in \mathcal{H}_t(\mathbb{R}^n)$ for any $t > 0$. Moreover, it can be shown that the functions $\tilde{\Phi}_\alpha(x) = e^{-(2|\alpha|+n)t}\Phi_\alpha(x)$ form an orthonormal system in $\mathcal{H}_t(\mathbb{R}^n)$.

Theorem 3.9. *The family* $\{\tilde{\Phi}_\alpha(x) : \alpha \in \mathbb{N}^n\}$ *is an orthonormal basis for* $\mathcal{H}_t(\mathbb{R}^n)$. *Equivalently,* $\{\tilde{\Phi}_\alpha(z) : \alpha \in \mathbb{N}^n\}$ *is an orthonormal basis for* $\mathcal{H}_t(\mathbb{C}^n)$.

We will not attempt a proof of this theorem but only indicate a major step involved in the proof. Before that let us see how this theorem can be used to prove the invariance of $\mathcal{H}_t(\mathbb{R}^n)$. Any $f \in \mathcal{H}_t(\mathbb{R}^n)$ has an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \tilde{\Phi}_\alpha$$

where the sequence c_α is square summable. It is now obvious that

$$\mathcal{F}f = \sum_{\alpha \in \mathbb{N}^n} (-i)^{|\alpha|} c_\alpha \tilde{\Phi}_\alpha$$

also belongs to $\mathcal{H}_t(\mathbb{R}^n)$.

Let $\mathcal{B}_t(\mathbb{C}^n)$ be the Bergman space consisting of all entire functions F that are square integrable with respect to the weight function

$$p_{t/2}(y) = t^{-\frac{n}{2}} e^{-\frac{\pi}{t}|y|^2}.$$

That is, $F \in \mathcal{B}_t(\mathbb{C}^n)$ if and only if $F(x+iy)$ is entire and

$$\|F\|^2_{\mathcal{B}_t(\mathbb{C}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x+iy)|^2 p_{t/2}(y) dx dy < \infty.$$

As before we let $\mathcal{B}_t(\mathbb{R}^n)$ stand for the subspace consisting of all restrictions of members of $\mathcal{B}_t(\mathbb{C}^n)$ to $\mathbb{R}^n$. We have the following characterisation of these Bergman spaces.

Theorem 3.10. *A function* $F \in \mathcal{B}_t(\mathbb{C}^n)$ *if and only if* $F = f * p_t$ *for some* $f \in L^2(\mathbb{R}^n)$. *Moreover,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x+iy)|^2 p_{t/2}(y) dx dy = \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

Proof. When $F = f * p_t$ so that $\mathcal{F}F(\xi) = \mathcal{F}f(\xi) e^{-2\pi t|\xi|^2}$ the inversion formula shows that the prescription

$$F(x+iy) = \int_{\mathbb{R}^n} \mathcal{F}f(\xi) e^{-2\pi t|\xi|^2} e^{2\pi i(x+iy)\cdot\xi} d\xi$$

gives an entire extension of F. Moreover, by Plancherel theorem

$$\int_{\mathbb{R}^n} |F(x+iy)|^2 dx = \int_{\mathbb{R}^n} |\mathcal{F}f(\xi)|^2 e^{-4\pi t|\xi|^2} e^{-4\pi y\cdot\xi} d\xi.$$

Integrating both sides against $p_{t/2}(y)$ and simplifying we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |F(x+iy)|^2 p_{t/2}(y)dxdy = \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

The converse is not so easy to prove. We need to introduce a related space through which we will obtain the converse. □

The space $\mathcal{B}_t(\mathbb{C}^n)$ was introduced and studied by Bargmann and Segal. Bargmann gave a direct proof of the above theorem but we wish to go through the Fock spaces which is more instructive. Let $\mathcal{F}_t(\mathbb{C}^n)$ be the space of all entire functions G for which

$$\|G\|^2_{\mathcal{F}_t(\mathbb{C}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |G(x+iy)|^2 e^{-\frac{1}{2t}\pi(|x|^2+|y|^2)} dxdy < \infty.$$

It is then easy to check that $F \in \mathcal{B}_t(\mathbb{C}^n)$ if and only if $G(z) = F(z)e^{\frac{1}{4t}z^2} \in \mathcal{F}_t(\mathbb{C}^n)$. In view of this we only need to prove the following.

Theorem 3.11. *If $G \in \mathcal{F}_t(\mathbb{C}^n)$ then there exists f in $L^2(\mathbb{R}^n)$ such that $G(x) = f * p_t(x)e^{\frac{1}{4t}\pi|x|^2}$.*

We prove the theorem after some preparation. Let $\mathcal{O}(\mathbb{C}^n)$ be the space of all entire functions equipped with the topology of uniform convergence over compact subsets of $\mathbb{C}^n$. We claim that the inclusion $\mathcal{F}_t(\mathbb{C}^n) \subset \mathcal{O}(\mathbb{C}^n)$ is continuous.

Lemma 3.12. *Given any compact $K \subset \mathbb{C}^n$ there exists a constant C_K such that*

$$\sup_{z\in K} |G(z)| \leq C_K \|G\|_{\mathcal{F}_t(\mathbb{C}^n)}, \;\; F \in \mathcal{F}_t(\mathbb{C}^n).$$

Proof. By mean value theorem for entire functions we have

$$G(z) = (\pi r^2)^{-n} \int_{D(z,r)} G(w)dw$$

where $D(z,r)$ is the polydisc $\{w \in C^n : |z_j - w_j| \leq r, j = 1, 2, ..., n\}$. By Cauchy-Schwarz inequality we get

$$|G(z)|^2 \leq (\pi r^2)_2 n e^{\frac{1}{2t}(|z|+r)^2} \|G\|^2_{\mathcal{F}_t(\mathbb{C}^n)}$$

from which we get the required estimate. □

Consider now the functions $\zeta^t_\alpha(z) = d_\alpha^{-1} z^\alpha$ where d_α are chosen so that

$$\int_{\mathbb{C}^n} |\zeta^t_\alpha(z)|^2 e^{-\frac{1}{2t}|z|^2} dz = 1.$$

As the measure $e^{-\frac{1}{2t}|z|^2} dz$ is radial it follows that $\{\zeta^t_\alpha : \alpha \in \mathbb{N}^n\}$ forms an orthonormal system in $\mathcal{F}_t(\mathbb{C}^n)$.

Lemma 3.13. *The system* $\{\zeta^t_\alpha : \alpha \in \mathbb{N}^n\}$ *is an orthonormal basis for* $\mathcal{F}_t(\mathbb{C}^n)$.

Proof. We first observe that $\mathcal{F}_t(\mathbb{C}^n)$ is complete. Indeed, if G_n is Cauchy in $\mathcal{F}_t(\mathbb{C}^n)$, then it converges to some $G \in L^2(\mathbb{C}^n, e^{-\frac{1}{2t}|z|^2} dz)$. In view of Lemma 3.12 G_n also converges uniformly over compact subsets. Hence the limit function G is entire which proves the completeness of $\mathcal{F}_t(\mathbb{C}^n)$.

Coming to the proof of the lemma, the Taylor expansion of any entire G can be written as

$$G(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha d_\alpha \zeta^t_\alpha(z)$$

where c_α are the Taylor coefficients of G. The series converges uniformly over compact subsets. As $G \in \mathcal{F}_t(\mathbb{C}^n)$, owing to the orthonormality of the system $\{\zeta^t_\alpha : \alpha \in \mathbb{N}^n\}$ the series also converges in $\mathcal{F}_t(\mathbb{C}^n)$ to some G_1. From the previous lemma, the series then converges to G_1 uniformly on compact subsets. This forces $G_1 = G$ and hence the lemma. □

We define another family of functions Φ^t_α on $\mathbb{R}^n$ by the prescription

$$\Phi^t_\alpha * p_t(x) = \zeta^t_\alpha(x) e^{-\frac{1}{4t}\pi|x|^2}.$$

It is easy to see that $\Phi_\alpha^t \in L^2(\mathbb{R}^n)$. As both sides of the defining relation above extend to $\mathbb{C}^n$ as entire functions it follows that

$$\Phi_\alpha^t * p_t(z) = \zeta_\alpha^t(z) e^{-\frac{1}{4t}\pi z^2}$$

for all $z \in \mathbb{C}^n$.

Lemma 3.14. *The system* $\{\Phi_\alpha^t : \alpha \in \mathbb{N}^n\}$ *is an orthonormal basis for* $L^2(\mathbb{R}^n)$.

Proof. We observe that

$$\int_{\mathbb{C}^n} \zeta_\alpha^t(z)\overline{\zeta_\beta^t(z)} e^{-\frac{1}{2t}\pi|z|^2} dz = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_\alpha^t * p_t(x+iy)\overline{\Phi_\beta^t * p_t(x+iy)} e^{-\frac{1}{t}\pi|y|^2} dx dy.$$

Using the result of Theorem we get

$$(\Phi_\alpha^t, \Phi_\beta^t) = (\zeta_\alpha^t, \zeta_\beta^t) = \delta_{\alpha,\beta}$$

which proves the orthonormality of the system. The completeness follows that of the system ζ_α^t. □

We are now in a position to prove Theorem. We write the Taylor expansion of G as

$$G(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \zeta_\alpha^t(z)$$

and define

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \Phi_\alpha^t.$$

Then clearly $f \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = \|G\|_{\mathcal{F}_t(\mathbb{C}^n)}$. Moreover, from the definition of Φ_α^t it follows that

$$f * p_t(z) = G(z) e^{-\frac{1}{4t}\pi z^2}.$$

Hence the theorem.

3.3. **Spherical harmonics.** In this subsection we look for some more eigenfunctions of the Fourier transform which have some symmetry. As in the one dimensional case we consider functions of the form $f(x) = p(x)e^{-\pi|x|^2}$. This will be an eigenfunction of $\mathcal{F}$ if and only p satisfies

$$\int_{\mathbb{R}^n} p(x - iy)e^{-\pi|x|^2} dx = \lambda p(y).$$

If this is true for all $y \in \mathbb{R}^n$ then we should also have

$$\int_{\mathbb{R}^n} p(x + y)e^{-\pi|x|^2} dx = \lambda p(iy).$$

Integrating in polar coordinates the integral on the left is

$$\int_0^\infty |S^{n-1}| \left(\int_{S^{n-1}} p(y + r\omega) d\sigma(\omega) \right) e^{-\pi r^2} r^{n-1} dr$$

where $d\sigma$ is the normalised surface measure on the unit sphere S^{n-1}. If p is homogeneous of degree m then $p(iy) = i^m p(y)$ and hence for such polynomilas the equation

$$\int_{\mathbb{R}^n} p(x + y)e^{-\pi|x|^2} dx = \lambda i^m p(y)$$

will be satisfied for $\lambda = (-i)^m$ if p has the mean value property

$$\int_{S^{n-1}} p(y + r\omega) d\sigma(\omega) = p(y).$$

Such functions are precisely the harmonic functions satisfying $\Delta u = 0$. Thus we have proved

Theorem 3.15. *Let $f(x) = p(x)e^{-\pi|x|^2}$ where p is homogeneous of degree m and harmonic. Then $\mathcal{F}f = (-i)^m f$.*

Let $\mathcal{P}_m$ stand for the finite dimensional space of homogeneous harmonic polynomilas of degree m. The above theorem says that the finite dimensional subspace of $L^2(\mathbb{R}^n)$ consisting of functions of the form $p(x)e^{-\pi|x|^2}, p \in \mathcal{P}_m$ is invariant under the Fourier transform. We claim that the following extension is true.

Theorem 3.16. *Let $f \in L^2(\mathbb{R}^n)$ be of the form $f(x) = p(x)g(|x|)$ where $p \in \mathcal{P}_m$. Then $\mathcal{F}f(\xi) = p(\xi)G(|\xi|)$. Thus the subspace of functions of* *the form* $f(x) = p(x)g(|x|), p \in \mathcal{P}_m$ *is invariant under the Fourier transform.*

Proof. When $f(x) = p(x)g(|x|), p \in \mathcal{P}_m$ is from L^2 the function g satisfies
$$\int_0^\infty |g(r)|^2 r^{n+2m-1} dr < \infty.$$
We let $\mathcal{D}_{n+2m}$ to stand for the space of all such functions with the obvious norm. We claim that the subspace W consisting of finite linear combinations of $e^{-\pi t|x|^2}$ as t runs through positive reals is dense in $\mathcal{D}_{n+2m}$. To see this suppose $g \in \mathcal{D}_{n+2m}$ satisfies
$$\int_0^\infty e^{-\pi t r^2} g(r) r^{n+2m-1} dr = 0$$
for all $t > 0$. Differentiating the integral k times at $t = 1$ we get
$$\int_0^\infty e^{-\pi r^2} r^{2k} g(r) r^{n+2m-1} dr = 0$$
for all $k \in \mathbb{N}$. Thus the function $g(r)r^{n+2m-1}e^{-\frac{1}{2}\pi r^2}$ is orthogonal to all functions of the form $P(r^2)e^{-\frac{1}{2}\pi r^2}$ where P runs through all polynomilas. As this is a dense class in $L^2((0,\infty), dr)$ we get $g = 0$.

In view of this density, it is enough to prove that W is invariant under Fourier transform. But this is easy to check. For $t > 0$ we let $f_t(x) = t^n f(tx)$ so that $\mathcal{F}(f_t)(\xi) = \mathcal{F}f(t^{-1}\xi)$. If $f(x) = p(x)e^{-\pi t^2|x|^2}$ take $g(x) = p(x)e^{-\pi|x|^2}$ and consider
$$\mathcal{F}(f)(\xi) = t^{-n-m}\mathcal{F}(g_t)(\xi) = t^{-n-m}\mathcal{F}(g)(t^{-1}\xi).$$
Since $\mathcal{F}g(\xi) = (-i)^m g(\xi)$ we get
$$\mathcal{F}(f)(\xi) = t^{-n-2m}(-i)^m p(x) e^{-\pi t^{-2}|x|^2}.$$
This proves that W is invariant and hence the theorem follows. □

The above theorem gives rise to an operator T_m^n on the space $\mathcal{D}_{n+2m}$ defined as follows. If $g \in \mathcal{D}_{n+2m}$ then for $p \in \mathcal{P}_m$ the function $p(x)g(|x|) \in L^2(\mathbb{R}^n)$ whose Fourier transform is of the form $p(x)G(|x|)$. As $\mathcal{F}$ is unitary it follows that $G \in \mathcal{D}_{n+2m}$. We define $T_m^n g = G$. Note that $\|T_m^n g\| = \|g\|$ where $\|\cdot\|$ is the norm in $\mathcal{D}_{n+2m}$. We can think of

$g(|x|)$ as a radial function on $\mathbb{R}^{n+2m}$ whose $n+2m$ dimensional Fourier transform will be a radial function, say $G_0(|x|)$. We define another operator T_0^{n+2m} on $\mathcal{D}_{n+2m}$ by letting $T_0^{n+2m}g = G_0$ It is also clear that $\|T_0^{n+2m}g\| = \|g\|$. If we denote the Fourier transform on $\mathbb{R}^n$ by $\mathcal{F}_n$ then $T_0^{n+2m} = \mathcal{F}_{n+2m}$. Calculations done in the proof of the above theorem shows that $T_m^n g = (-i)^m T_0^{n+2m} g$ whenever $g \in W$. The density of this subspace gives

Theorem 3.17. *Let $f \in L^2(\mathbb{R}^n)$ be of the form $f(x) = p(x)g(|x|), p \in \mathcal{P}_m$. Then $\mathcal{F}_n(f) = (-i)^m p \mathcal{F}_{n+2m} g$.*

The above result is known as the Hecke-Bochner formula for the Fourier transform. We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. Let J_α stand for the Bessel function of type $\alpha > -1$.

Theorem 3.18. *If $f(x) = g(|x|)$ is radial and integrable then*

$$\mathcal{F}_n(f)(\xi) = c_n \int_0^\infty g(r) \frac{J_{\frac{n}{2}-1}(2\pi r|\xi|)}{(2\pi r|\xi|)^{\frac{n}{2}-1}} r^{n-1} dr.$$

Proof. As f is radial

$$\mathcal{F}_n(f)(\xi) = |S^{n-1}| \int_0^\infty g(r) \left(\int_{S^{n-1}} e^{-2\pi i r \omega \cdot \xi} d\sigma(\omega) \right) r^{n-1} dr.$$

The inner integral is clearly a radial function as the measure $d\sigma$ is rotation invariant. It can be shown that the inner integral is a constant multiple of $\frac{J_{\frac{n}{2}-1}(2\pi r|\xi|)}{(2\pi r|\xi|)^{\frac{n}{2}-1}}$. This completes the proof. □

3.4. **Ultravariant subspaces of $L^2(\mathbb{R}^n)$.** In the previous section we studied several subspaces of $L^2(\mathbb{R}^n)$ that are invariant under $\mathcal{F}$. But not all subspaces are invariant. For example, $L^1 \cap L^2(\mathbb{R}^n)$ is not invariant under $\mathcal{F}$. In this section we are interested in subspaces which are extremely sensitive to the action of the Fourier transform. We make this precise in the following definition.

Definition 3.19. *We say that a subspace W of $L^2(\mathbb{R}^n)$ is ultravariant if the conditions $f \in W, \mathcal{F}f \in W$ imply $f = 0$.*

A priori it is not clear if there is any ultravariant subspace of $L^2(\mathbb{R}^n)$ but in this section we show that there are many such subspaces.

Our first example of an ultravariant subspace is the Paley-Wiener space defined as follows. For each $a > 0$ let $PW_a(\mathbb{R}^n)$ stand for the subspace of $L^2(\mathbb{R}^n)$ consisting of functions having entire extensions to $\mathbb{C}^n$ and satisfying

$$\int_{\mathbb{R}^n} |f(x+iy)|^2 dx \leq Ce^{4\pi a|y|}$$

for all $y \in \mathbb{R}^n$. We define $PW(\mathbb{R}^n) = \cup_{a>0} PW_a(\mathbb{R}^n)$ and call it the Paley-Wiener space. The space $PW(\mathbb{R}^n)$ is not empty since any $f \in L^2(\mathbb{R}^n)$ whose Fourier tranform is compactly supported belongs to the Paley-Wiener space. To see this, suppose $\mathcal{F}f$ vanishes for $|\xi| > a$ and consider the inversion formula

$$f(x) = \int_{|\xi|\leq a} \mathcal{F}f(\xi)e^{2\pi i x\cdot\xi} d\xi.$$

It is clear then that the prescription

$$f(x+iy) = \int_{|\xi|\leq a} \mathcal{F}f(\xi)e^{2\pi i(x+iy)\cdot\xi} d\xi$$

defines and entire function and by Plancherel we also have

$$\int_{\mathbb{R}^n} |f(x+iy)|^2 dx \leq Ce^{4\pi a|y|}.$$

We show below that the converse is also true.

Theorem 3.20. *An L^2 function f belongs to $PW_a(\mathbb{R}^n)$ if and only if $\mathcal{F}f$ is supported in $\{\xi : |\xi| \leq a\}$.*

Proof. It is enough to show that $\mathcal{F}f$ is compactly supported in $\{\xi : |\xi| \leq a\}$ whenever $f \in PW_a(\mathbb{R}^n)$ since the converse has been already proved. First we claim that $PW(\mathbb{R}^n) \subset \cap_{t>0}\mathcal{B}_t(\mathbb{R}^n)$. To see this let $f \in PW_a(\mathbb{R}^n)$ and consider

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x+iy)|^2 p_{t/2}(y) dx dy \leq Ct^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{4\pi a|y|} e^{-\frac{\pi}{t}|y|^2} dy.$$

A simple calculation shows that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x+iy)|^2 p_{t/2}(y) dx dy \leq C(a^2 t)^{\frac{n}{2}} e^{4\pi a^2 t}.$$

This proves our claim. In view of Theorem 2.5 we get $g_t \in L^2(\mathbb{R}^n)$ such that $f = g_t * p_t$ and

$$\int_{\mathbb{R}^n} |g_t(x)|^2 dx = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x+iy)|^2 p_{t/2}(y) dx dy.$$

For each $\delta > 0$ consider

$$\int_{|\xi|>a+\delta} |\hat{f}(\xi)|^2 d\xi \leq e^{-4\pi t(a+\delta)^2} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 e^{4\pi t|\xi|^2} d\xi.$$

As $f = g_t * p_t$ the last integral is equal to

$$\int_{\mathbb{R}^n} |g_t(x)|^2 dx = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x+iy)|^2 p_{t/2}(y) dx dy.$$

This along with our earlier estimate gives

$$\int_{|\xi|>a+\delta} |\hat{f}(\xi)|^2 d\xi \leq C(a^2 t)^{\frac{n}{2}} e^{-4\pi t(a+\delta)^2} e^{4\pi t a^2}.$$

Letting t go to infinity we conclude that $\hat{f}$ vanishes for $|\xi| > a + \delta$. Since δ is arbitrary we see that $\hat{f}$ is supported in $|\xi| \leq a$. □

Theorem 3.21. *The Paley-Wiener space $PW(\mathbb{R}^n)$ is ultravariant.*

The thorem follows immediately from the Paley-Wiener theorem. If $f \in PW(\mathbb{R}^n)$ then $\hat{f}$ is compactly supported and hence cannot have an entire extension unless $f = 0$.

The Paley-Wiener space is strictly contained in $\cap_{t>0}\mathcal{B}_t(\mathbb{R}^n)$. It turns out that the bigger space $\cap_{t>0}\mathcal{B}_t(\mathbb{R}^n)$ is also ultravariant. Even more surprising is the following result. Note that the heat kernel $p_s \in \mathcal{B}_t(\mathbb{R}^n)$ only for $t < s$ since

$$\mathcal{F}p_s(x) = e^{-2\pi s|x|^2} = (2s)^{-\frac{n}{2}} p_{\frac{1}{4s}}(x).$$

It is also clear that $\mathcal{F}p_s \in \mathcal{B}_t(\mathbb{R}^n)$ only for $t < \frac{1}{4s}$. Therefore, if $0 < t < 1/2$ then for any s satisfying $t < s < 1/4t$ the function p_s and $\mathcal{F}p_s$ both belong to $\mathcal{B}_t(\mathbb{R}^n)$. This means that for such t the space $\mathcal{B}_t(\mathbb{R}^n)$ is not ultravariant. But the behaviour is different for other values of t.

Theorem 3.22. *The Bergman space $\mathcal{B}_t(\mathbb{R}^n)$ is ultravariant for all $t \geq 1/2$.*

This is a very special case of a theorem of Cowling and Price which is viewed as an uncertainty principle for the Fourier transform. There is a more general theorem due to Bonami et al from which Cowling-Price theorem can be deduced. We just state the result without proof.

Theorem 3.23. *The only function $f \in L^2(\mathbb{R}^n)$ for which*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||\mathcal{F}f(y)|e^{2\pi|x\cdot y|}dxdy < \infty$$

is the trivial function $f = 0$.

In the one dimensional case this result is due to Beurling. Let us use this to prove the ultravariance of $\mathcal{B}_t(\mathbb{R}^n)$ for $t > 1/2$. If both f and $\mathcal{F}f$ are in $\mathcal{B}_t(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |g(x)|^2 e^{4\pi t|x|^2}dx < \infty$$

for $g = f$ as well as for $g = \mathcal{F}f$. It is then easy to check that the hypothesis of Beurling's theorem is satisfied.

Another family of ultravariant subspaces is provided by the Hardy classes. For $t > 0$ we define $H_t(\mathbb{R}^n)$ to be the subspace of functions satisfying the estimate $|f(x)| \leq Cp(x)e^{-\pi t|x|^2}$ for some polynomial p.

Theorem 3.24. *The Hardy class $H_t(\mathbb{R}^n)$ is ultravariant for all $t > 1$.*

Since the Hermite functions $\Phi_\alpha \in H_t(\mathbb{R}^n)$ for all $t \leq 1$ it follows that $H_t(\mathbb{R}^n)$ is not ultravariant for $t \leq 1$. We do not know if it is invariant or not. The above theorem follows from Hardy's uncertainty principle which we proved in the one dimensional case earlier. Here we prove it on $\mathbb{R}^n$ in a slightly stronger form which is needed for the proof of the above theorem.

Theorem 3.25. *Let $f \in L^2(\mathbb{R}^n)$ satisfy the estimates*

$$|f(x)| \leq Cp(x)e^{-\pi t|x|^2}, \ |\mathcal{F}f(y)| \leq Cq(y)e^{-\pi s|x|^2}$$

for some $s, t > 0$ and p, q polynomials. Then $f = 0$ whenever $st > 1$ and $f(x) = Cp(x)e^{-\pi t|x|^2}$ when $st = 1$.

Proof. First we consider the case $st > 1$ in one dimension. We can choose $\epsilon, \delta > 0$ such that $(s-\delta)(t-\epsilon) = 1$. Then we have the estimates

$$|f(x)| \leq C_\epsilon e^{-\pi(t-\epsilon)x^2}, \ |\mathcal{F}f(y)| \leq C_\delta e^{-\pi(s-\delta)y^2}.$$

By the previous theorem we conclude that $f(x) = Ce^{-\pi(t-\epsilon)x^2}$. But this cannot satisfy the hypothesis unless $C = 0$ proving that $f = 0$.

To prove the theorem in n dimensions, we fix a vector $\omega \in S^{n-1}$ and consider the function defined on $\mathbb{R}$ by

$$f_\omega(v) = \int_{\mathbb{R}^{n-1}} f(u \oplus v\omega)du.$$

Then it is easy to see that

$$\int_{\mathbb{R}} f_\omega(v)e^{-2\pi ivr}dv = \hat{f}(r\omega).$$

Thus the function f_ω and its Fourier transform satisfy the hypothesis with $st > 1$ and hence $\hat{f}(r\omega) = 0$. As this is true for all ω we conclude that $f = 0$. To prove the equality case $st = 1$ with polynomial factors, again we need to consider the one dimensional case. The proof of Theorem can be modified to take care of this case. The details are left to the reader. □