THE UNIFORMISATION THEOREM OF RIEMANN SURFACES

1. WHAT IS THE AIM OF THIS SEMINAR?

Recall that a compact oriented surface is a "g"-holed object. (Classification of surfaces.) It can be obtained through a 4g sided polygon by identification of alternate sides. Its fundamental group is generated by g generators $A_1, A_2, \ldots, A_g$ with the only relation being $\Pi_i[A_i, B_i] = 1$.

With this information, can we answer questions like is so and so group a subgroup of the fundamental group? Normally, to study groups, it is profitable to study their matrix representations, i.e., to think of homomorphisms from $G$ to $\text{Mat}$ where $\text{Mat}$ is some matrix group like $\text{SL}(2, \mathbb{R})$ for example. Shockingly enough, it turns out that many representations of the fundamental group of surfaces correspond to geometric structures on the surface like "good metrics" or "nice vector bundles", etc. The aim of this seminar is to study such correspondences between representations of the fundamental groups of surfaces and geometric structures.

2. WHAT IS THE AIM OF RIEMANNIAN GEOMETRY?

Recall that a symmetric 2-tensor $g$ is a Riemannian metric on a manifold $M$ if locally $g$ is simply a symmetric positive-definite matrix. The ideal aim of Riemannian geometry is to write down a list of standard Riemannian manifolds so that every Riemannian manifold is isometric (i.e. there is a diffeomorphism relating the two metrics) to one of the elements of the list. This is of course too much to ask. What one can at least do is to ask "Is there a "best" choice of a metric on a given Riemannian manifold? That is, can we index Riemannian manifolds by judiciously chosen metrics?"

Let us look at the simplest non-trivial example of surfaces. What should the notion of "best metric" be? The obvious answer is "One that is similarly curved everywhere", i.e., where the curvature (there is only one notion of curvature on a surface, namely, the Gaussian curvature. Recall that the Gaussian curvature at a point is obtained by choosing two curves whose "curvature" is the maximum and minimum and multiplying their curvatures) is a constant. So how does one come up with constant curvature metrics on a Riemannian surface?

3. THE RIEMANNIAN UNIFORMISATION THEOREM

The curvature $K$ depends on the second derivatives of a metric. So saying $K = \text{constant}$ is asking us to solve a PDE for $g$. It is easier to solve an equation with one unknown than with several (Of course, even in linear algebra, we have a formula for the former but not for the latter). So given a single function $f$ and a metric $g_0$, how do you come up with a new metric? Of course, you rescale the old one to $g = e^{-f}g_0$. Computing the new curvature we get (See list of formulas in Riemannian geometry on wikipedia to get the correct formula),

$$\Delta f = Ke^{-f} - K_0,$$

(3.1)

where $K_0$ is the Gaussian curvature of $g_0$, $K$ is the new curvature, $\Delta f$ is locally, at a point where we choose coordinates such that $g_0(p)$ is the Euclidean metric up to the second order, $\Delta f(p) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. 

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The question is - Can we solve this equation? If so, is the solution unique? The answer (which is supposedly blowing in the wind) is provided by the Riemannian uniformisation theorem -

**Theorem 3.1.** In every conformal class of metrics \([g]\) on a compact oriented surface, there exists a unique (up to rescalings by positive constants) metric of constant curvature.

**Proof.** It is actually quite hard to prove (shockingly enough) this theorem for genus \(g = 0\), i.e., for a sphere! (Of course there is one metric of positive constant curvature that even children (who do not believe the flat-earth theory) know about. The issue is that are there other conformal classes? (There aren’t) If there are, how do you prove that they have such metrics? The technique we are going to describe below will run into serious challenges for \(g = 0\).) In fact, this is no coincidence. It turns out that one generalisation of this observation has been proven recently by Chen-Donaldson-Sun (and apparently independently by Tian). It is called the Yau-Tian-Donaldson conjecture. Another generalisation called the Yamabe problem was solved earlier.

Let us take the next case of \(g = 1\). Note that by the Gauss-Bonnet theorem, \(\int KdA = \int_M K_0 dA_0 = 2\pi(2 - 2g) = 0\). Therefore we want \(K = 0\). This means we have to solve

\[ \Delta f = -K_0. \]

Let’s prove uniqueness first. Indeed, if \(f_1, f_2\) are two solutions, then \(\Delta(f_1 - f_2) = 0\). Multiplying by \(f_1 - f_2\) and integrating-by-parts we get

\[ -\int_M |\nabla(f_1 - f_2)|^2 = 0. \]

Therefore \(f_1 - f_2\) is a constant. The point of this calculation is “If you want to prove that the kernel of some operator is trivial, multiply by something and integrate-by-parts”. If this was linear algebra, then we would say “A linear map \(T\) is surjective if and only if the codomain is orthogonal to the kernel of \(T^\dagger\)”. But we are dealing with infinite-dimensional vector spaces here. Fortunately, the Laplacian is well-behaved (the so-called “elliptic” operators like the Laplacian obey the so-called Fredholm’s alternative) and all your intuition from linear algebra goes through. Actually, since we will require the basic idea later on (in Donaldson’s proof of Narasimhan and Seshadri’s theorem), here is a sketch:

Consider the “energy” (it is honestly the electrostatic energy) \(E = \frac{1}{2} \int_M \nabla f^2 + \int_M K_0 f\). The claim is that the minimum of this energy (over all “nice” functions) is attained at a smooth function \(f\) and that indeed the minimum does satisfy the PDE. Let’s prove the latter first. Indeed, since \(E(f)\) is the minimum, then \(\frac{dE(f+tv)}{dt} = 0\) at \(t = 0\) for any smooth function \(v\). Therefore,

\[ \int_M <\nabla f, \nabla v> + \int_M K_0 v = 0 \]

\[ \Rightarrow -\int_M \Delta f v + \int_M K_0 v = 0 \ \forall \ v. \]  

Choosing \(v = K_0 - \Delta f\) we see that indeed \(\Delta f = K_0\).

So we just have to “just” prove that the minimum over all smooth functions satisfying \(\int_M f dA_0 = 0\) is attained. Is this energy even bounded below? For that we have an inequality (called the Poincaré inequality):
For smooth functions of zero average, \( \int_M |\nabla f|^2 \geq C \int_M |f|^2 \). An easy application of the Cauchy-Schwarz inequality and the Poincare inequality will allow us to conclude that indeed \( E \) is at least bounded. So there is a sequence \( f_n \) of functions such that \( E[f_n] \) converges to inf \( E \). So all of these energies are bounded. Ideally, we would like to say that \( E[f_n] \to \inf E \). So all of these energies are bounded. Firstly, this may not be true for trivial reasons, in that perhaps only a subsequence might converge. But even with that, how can we get a “compactness” result?

That is, does a bounded sequence in the so-called Sobolev norm \( \|u\|_{H^1}^2 = \int_M |\nabla u|^2 dA_0 + \int_M u^2 dA_0 \) have a convergent subsequence? Firstly, the so-called Sobolev space \( H^1 \) (defined as the completion within \( L^2 \) of the smooth functions under the Sobolev norm) is a Hilbert space containing smooth functions as a dense subspace. Secondly, the answer is NO. (The unit ball of a Hilbert space is compact if and only if it is finite-dimensional → Not so easy to prove.) However, thanks to the scary sounding Banach-Alaoglu theorem, a subsequence converges weakly to a function \( f \). (That is \( \int \phi v = \lim \int \phi_n v \forall v \).) Also, \( E[f] \leq \lim E[f_n] \) and is therefore equal (because it is the infimum).

The only thing remaining is to show that if \( f \in H^1 \) is the minimum, then it is actually a smooth function. This is the hardest part of the proof. We will skip it for now.

So much for \( g = 1 \). For higher genus, we want \( K < 0 \). Now we are faced with a nonlinear PDE. Here is a beautiful method (originally due to Bernstein) to handle such PDE. It is called the method of continuity. Consider the following family of PDE indexed by a number \( 0 \leq t \leq 1 \).

\[
\Delta f_t = -e^{-f_t} - tK_0 + (1 - t).
\] (3.3)

At \( t = 0 \) there is obviously a solution \( \phi_0 = 0 \). If we prove that the set of \( t \) for which there exists a smooth solution is both open and closed, then by connectedness, the set is \([0, 1] \).

(1) Openness: Basically, given a solution at \( t = t_0 \), we need to prove that there are solutions nearby. Consider the following map,

\[
T(t, f) = \Delta f + e^{-f} + tK_0 - (1 - t).
\] (3.4)

Naively speaking, if this was a map between finite dimensional things, then by implicit function theorem, if its derivative with respect to \( f \) is surjective then we will be done. Indeed, there is an implicit function theorem on Banach spaces. The appropriate Banach spaces to consider are \( \mathbb{R} \times C^{k+2,\alpha} \) and \( C^{k,\alpha} \) for a given integers \( k \geq 0 \) and \( \alpha > 0 \). Why the \( \alpha \)? (H"older space) It is for a technical reason as we shall see in a moment. The “derivative” with respect to \( f \) being surjective is the same as saying, for every \( v \in C^{k,\alpha} \) there exists a \( u \int C^{k+2,\alpha} \) such that

\[
\frac{d}{ds} \big|_{s=0} T(t_0, f_0 + su) = v
\] \[
\Rightarrow \Delta u - e^{-f_0} u = v.
\] (3.5)

Borrowing from our intuition from linear algebra (it is easy to verify that the above equation for \( u \) is self-adjoint), i.e., using the Fredholm alternative, we simply need to show that the
kernel is trivial. Indeed if \( u \) is in the kernel, then

\[
\Delta u = e^{-f_0}u \\
\Rightarrow - \int_M |\nabla u|^2 \, dA_0 = \int_M u^2 e^{-f_0} \, dA_0,
\]

which means that \( u = 0 \).

(2) Closedness : This is usually the harder part of any method of continuity. What does it mean for the set to be closed? It means that for any sequence \( t_n \to t \) such that \( f_{t_n} \) exist, there exists a solution \( f_t \) at \( t \). In other words, if we can prove that a subsequence \( f_{t_{n_k}} \to f_t \) in \( C^{2,\alpha} \) then we will be done. Beautifully enough, the Arzela-Ascoli theorem implies that (do this as an exercise) if \( \beta > \alpha \) and a sequence \( w_n \) is bounded independent of \( n \) in \( C^{2,\beta} \), then a subsequence converges in \( C^{2,\alpha} \). Thus, to show closedness, it is enough to prove that solutions to equation 3.3 have a uniform \( C^{2,\beta} \) estimate independent of \( t \).

Indeed, such estimates are proven by improving upon lower order estimates -

Let’s see if we can at least prove that \( \|f_t\|_{C^0} \leq C \). Indeed, at the maximum of \( f_t \), easy calculus shows that \( \Delta f_t \leq 0 \). (Second derivative test.) Therefore \( -e^{-f_t(\text{max})} - tK_0 + (1 - t) \leq 0 \). This means that \( f_t(\text{max}) \leq C \). Likewise \( f_t(\text{min}) \geq c \).

Actually, now we have some standard results in PDE theory (read Kazdan’s notes for instance) that say effectively the following: If the right hand side of \( \Delta f = h \) is bounded in \( L^p \) for all large \( p \), then \( f \) is actually bounded in \( C^{1,\alpha} \) for some \( \alpha > 0 \) (If you want to sound cool, it is called \( L^p \) regularity + Sobolev embedding). There is another result (Schauder’s estimates) that implies that if the right hand side of \( \Delta f = h \) is bounded in \( C^{0,\alpha} \) and \( \|f\|_{C^0} \leq C \), then actually \( \|f\|_{C^{2,\alpha}} \leq C \). So combining all of these, we get our desired estimates. (These are called “a priori” estimates.)

As for uniqueness, suppose \( f_1, f_2 \) satisfy the equation for \( K < 0 \). Then

\[
\Delta(f_1 - f_2) = K(e^{-f_1} - e^{-f_2}) \\
\Rightarrow - \int_M |\nabla (f_1 - f_2)|^2 = K \int_M (f_1 - f_2)(e^{-f_1} - e^{-f_2}).
\]

(3.6)

This means that \( f_1 - f_2 \) is a constant.

Actually, uniqueness is quite easy for all three cases \( K = 0, > 0, < 0 \) assuming the Killing-Hopf theorem of the next section.

By the way, for \( K > 0 \), here is a way to prove some things: Firstly, in the conformal class of the usual round metric, there exists a constant curvature metric (the round one). Then assuming one knows complex geometry one proves that there is only one complex structure on the sphere. (This involves a little bit of algebraic geometry.) Thus there is only one conformal class and we are done.
4. Killing-Hopf theorem

So what if we find a constant Gaussian curvature metric on a surface? Big deal! Actually, there is an old theorem called the Killing-Hopf theorem that implies (in the special case of surfaces) that a constantly curved surface is isometric to a quotient of one of the following:

1. \( \mathbb{R}^2 \) with the Euclidean metric. (Flat earth according to some idiots.)
2. \( S^2 \) with the standard round metric.
3. \( \mathbb{H}^2 \), i.e. the upper half-plane with the metric \( g = \frac{dx^2 + dy^2}{y^2} \).

In other words, given a conformal class of metrics \([g]\) on a compact oriented surface, there is a unique representative in the conformal class of unit volume such that it is isometric to a quotient of one of the things above. In other words, if you care only about measuring angles (and not distances), you are always a quotient of the standard spaces. Already it looks like complex analysis might play a role here. (Recall that a biholomorphism preserves angles, and vice-versa in the complex plane.)

5. Complex manifolds

Seeing that we are quickly entering complex analysis, let us define complex manifolds. A complex manifold \( M \) of dimension \( n \) is a smooth manifold of dimension \( 2n \) such that it is locally diffeomorphic to an open set of \( \mathbb{C}^n \) and such that the transition maps are biholomorphisms.

Hold on! What is the meaning of a holomorphic map from \( \mathbb{C}^n \) to \( \mathbb{C} \)? It is simply holomorphic on each of the coordinates, i.e., the complex partial derivatives exist.

Anyway, the simplest complex manifolds are those of dimension 1. They are called Riemann surfaces. What are examples of Riemann surfaces? Well \( \mathbb{C} \) is one. Any quotient of \( \mathbb{C} \) by a lattice, i.e., a torus \( \mathbb{C}/\mathbb{Z}^2 \) is one. A famous one is \( \mathbb{C}\mathbb{P}^1 \), i.e., \( \mathbb{C}^2 - \{(0,0)\} \) quotiented out by : \( (X_0, X_1) \) identified with \( \lambda(X_0, X_1) \) where \( \lambda \neq 0 \) is complex. It is not hard to see that this is the same as the sphere. The upper half-plane is one such example. Actually, it is a nice exercise to prove that any isometry of the upper half-plane is actually a biholomorphism. Another nice exercise is to show that every complex manifold is orientable. (Just calculate the Jacobian and you will see....) A third nice exercise is to show that if you quotient out a complex manifold by a group of biholomorphisms in such a way that the quotient is a smooth manifold, then the quotient is also a complex manifold.

So, the Riemannian uniformisation theorem implies that given a conformal class of metrics \([g]\) on a compact oriented surface \( M \) of genus \( \geq 1 \), one can treat \( (M, [g]) \) as a Riemann surface by simply treating the manifold as a quotient of either the plane or the upper half-plane. The real question is, does every compact Riemann surface of genus \( \geq 1 \) arise this way and is every genus 0 compact Riemann surface always \( \mathbb{C}\mathbb{P}^1 \)?

6. The uniformisation theorem for Riemann surfaces

The answer to the previous question is yes. Indeed,

**Theorem 6.1.** Every Riemann surface (even the noncompact ones) arises as a quotient of the plane, the upper half-plane, or \( \mathbb{C}\mathbb{P}^1 \).

We can prove this for compact Riemann surfaces using the Riemannian uniformisation theorem. All we need to do is somehow relate conformal classes of metrics to complex structures. To do this,
we need to understand a simple question:
In what way can we relate \( \mathbb{C} \) and \( \mathbb{R}^2 \)?
This question sounds silly, but what we mean is the following - If I give you \( \mathbb{R}^2 \), what information would you need to call it \( \mathbb{C} \)? You would have to somehow make it a complex vector space. So you would need to know what multiplication of \((a, b)\) with \(\sqrt{-1}\) means. Indeed, usually, \(z = x + \sqrt{-1}y\) and therefore \(\sqrt{-1}z = \sqrt{-1}x - y\), i.e., \(\sqrt{-1}(1, 0) = (0, 1)\) and \(\sqrt{-1}(0, 1) = (-1, 0)\). So multiplication by \(\sqrt{-1}\) is simply a linear map \(J : \mathbb{R}^2 \to \mathbb{R}^2\) such that \(J^2 = -I\). (Exercise: Prove that if \(J : V \to V\) where \(V\) is a real vector space is such that \(J^2 = -I\), then indeed \(V\) is \(2n\) dimensional and that there is a real basis \(e_1, w_1, e_2, w_2, \ldots\) of \(V\) such that \(Je_i = w_j, Jw_i = -e_i\).) This is known as an almost complex structure.

So naively speaking, if we have a linear map \(J\) from the tangent bundle of a surface \(M\) to itself such that \(J^2 = -I\), then we can treat \(M\) as a complex manifold such that locally indeed there exist coordinates \((x, y)\) so that \(J \frac{\partial}{\partial x} = \frac{\partial}{\partial y}\) and \(J \frac{\partial}{\partial y} = -\frac{\partial}{\partial x}\)? The answer is yes. But in higher dimensions, it is NO. (There is an additional condition called Integrability. It is a deep theorem called the Newlander-Nirenberg theorem.)

Anyway, what does all of this have to do with what we were discussing? If you give me a conformal class of metrics, I can come up with an almost complex structure. Indeed, \(J\) is simply “rotate “anticlockwise” (with respect to the given orientation) by ninety degrees”. Likewise, if you give me a Riemann surface, then here is a conformal class of metrics: Choose any Hermitian metric \(h\) on the complex tangent bundle (spanned locally by \(\frac{\partial}{\partial z}\)). This defines a Riemannian metric if you identify the complex tangent bundle with the real one as above. You can even prove that an isometry of a metric induces a biholomorphism between the corresponding Riemann surfaces. This along with the Riemannian uniformisation theorem proves the Riemann surface uniformisation theorem.