

# COMPLEX GEOMETRY OF TEICHMÜLLER DOMAINS

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ABSTRACT. We say that a domain  $\Omega \subset \mathbb{C}^N$  is a *Teichmüller domain* if  $\Omega$  is biholomorphic to the Teichmüller space of a surface of finite type. In this survey we discuss the classical construction of such a domain due to Bers, outline what is known of its structure, and subsequently focus on recent developments concerning the Euclidean convexity of Teichmüller domains.

## 1. INTRODUCTION

The complex analytical theory of Teichmüller spaces was initiated in the early works of Teichmüller (see for example [2]) but it was not until the work of Ahlfors–Bers in the 1960s that they were realized, and studied, as bounded domains in finite-dimensional complex Euclidean spaces (see [8]).

In this survey we shall assume that the complex structure of Teichmüller space is acquired through the *Bers embedding* that we describe in §3; for more intrinsic characterizations see, for example, [39] or [16]. The biholomorphism type of the image Bers domain, or the nature of their boundaries is still mysterious, and we shall mention some open questions. Throughout, we consider the case of Riemann surfaces of finite analytical type (that is, having finite genus and finitely many punctures); see [14] for the case of surfaces of infinite type.

More generally, we define a *Teichmüller domain* to be an open connected set in a complex Euclidean space that is biholomorphic to the Teichmüller space of a Riemann surface of finite type. It was known that Teichmüller domains are pseudoconvex, which is a weaker notion than the usual notion of Euclidean convexity. Until recently, it had been an open problem if Teichmüller domains of more than one complex dimension, could also be Euclidean convex. What was intriguing was their similarity to such convex domains, a striking manifestation of which is the existence of complex geodesics – we refer the reader to the survey article of M. Abate and G. Patrizio [1] for a systematic study of this aspect of the geometry of Teichmüller domains. In this article we shall elaborate on the recent results of V. Markovic and the authors that established some *non*-convex features of Teichmüller domains.

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An important tool for probing the structure of a bounded domain is an *invariant metric*, i.e., a biholomorphism-invariant Finsler metric on the domain. While there are several such metrics which are important in different contexts, the Kobayashi and Carathéodory metrics are particularly relevant to the study of convex domains. This is highlighted in the fundamental work of L. Lempert, who proved that these metrics coincide if the domain is convex. We discuss two recent developments, in the setting of Teichmüller domains, regarding these issues: the first is the result of Markovic [31] that the Kobayashi and Carathéodory metrics on the Teichmüller space of a closed surface of genus greater than one, are not equal. Hence, by the aforementioned result of Lempert, such a Teichmüller domain is never convex. Second, we give a brief exposition of the main theorem of [18] asserting that, a Teichmüller domain cannot even be locally strictly convex at any boundary point.

It would be interesting to investigate whether the techniques in these papers can be used to ascertain the finer structure of the boundary of a Teichmüller domain.

**Plan of the article.** In §2 we recall some background relevant to this survey, and in §3 describe the Bers embedding of Teichmüller space, summarize what is known about the structure of the resulting Bers domain, and mention some open questions. In §4 we introduce the Kobayashi, Carathéodory and Teichmüller metrics and the notion of complex geodesics and Teichmüller disks. In §5, we discuss results asserting that Teichmüller are not biholomorphic to product complex manifolds. In §6, we survey some results about pseudoconvexity and convexity of Teichmüller spaces. The concluding sections (§7 and §8) are expositions of works of Markovic and Gupta–Seshadri.

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## 2. PRELIMINARIES

We start with some basic definitions. We refer the reader to [3] or [19] for an exposition of the following notions.

Let  $X$  be a Riemann surface of genus  $g$  and  $n$  punctures such that  $2g - 2 + n > 0$ . Let  $K_X$  denote the canonical bundle of  $X$ .

A *holomorphic quadratic differential* on  $X$  is a holomorphic section of  $K_X \otimes K_X$ . Let  $\mathcal{Q}(X)$  be the space of holomorphic quadratic differentials on  $X$ .

A *Beltrami differential* on  $X$  is a  $L^\infty$ -section of  $K_X^{-1} \otimes \overline{K}_X$ . We denote the space of Beltrami differentials on  $X$  by  $\mathcal{BD}(X)$ . In a holomorphic chart  $U \subset X$ , an element of  $\mathcal{BD}(X)$  has the form

$$\mu \frac{d\bar{z}}{dz}$$

where  $\mu \in L^\infty(U)$  is called a *Beltrami coefficient*.

By the Uniformization Theorem, the universal cover of the surface  $X$  is the upper half plane  $\mathbb{H}$ , and we have

$$X = \mathbb{H}/\Gamma$$

for  $\Gamma$  a *Fuchsian group*, namely a discrete subgroup of  $\text{Aut}(\mathbb{H}) = \text{PSL}_2(\mathbb{R})$ .

A *quasiconformal map* between two Riemann surfaces  $X$  and  $Y$  is a homeomorphism  $f : X \rightarrow Y$  with weak partial derivatives (in the sense of distributions) that are locally square-integrable, such that the Beltrami coefficient

$$(1) \quad \mu = \frac{f_{\bar{z}}}{f_z}$$

satisfies  $\|\mu\|_\infty < 1$ . Such a map lifts to a quasiconformal map

$$\tilde{f} : \mathbb{H} \rightarrow \mathbb{H}$$

that extends to a homeomorphism from  $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$  to itself. The corresponding Beltrami coefficient  $\tilde{\mu}$  then lies in the open unit ball of the complex Banach space of bounded Beltrami coefficients

$$\mathcal{BD}_\Gamma(\mathbb{H}) = \{\tilde{\mu} \in L^\infty(\mathbb{H}) : \tilde{\mu} \text{ is } \Gamma\text{-invariant}\}$$

From this perspective, one can define Teichmüller space by considering quasiconformal maps from a fixed basepoint, up to homotopy (or isotopy) of the maps relative to the punctures:

$\mathcal{T}_{g,n} = \{(Y, k) : Y \text{ is a Riemann surface, } k : X \rightarrow Y \text{ is quasiconformal map}\} / \sim$   
where  $(Y, k) \sim (Z, h)$  if and only if there is a *conformal* homeomorphism  $c : Y \rightarrow Z$  such that the composition

$$f = h^{-1} \circ c \circ k : X \rightarrow X$$

is a quasiconformal map that lifts to a map  $\tilde{f} : \mathbb{H} \rightarrow \mathbb{H}$  that extends to the identity map on  $\mathbb{R}$ . Since  $X$  is of finite analytical type, the last condition is equivalent to the map  $f$  being homotopic to the identity map.

The foundation for the complex-analytic theory of Teichmüller spaces is the Measurable Riemann Mapping Theorem (see [3] for an exposition):

**Theorem 2.1** (Ahlfors–Bers–Morrey). *Let  $\hat{\mathbb{C}}$  be the Riemann sphere and let  $\mu \in L^\infty(\hat{\mathbb{C}})$  satisfy  $\|\mu\|_\infty < 1$ . Then there is a quasiconformal map*

$$f^\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

*with Beltrami coefficient  $\mu$ , that is, satisfying the Beltrami equation (1). Such a solution is unique up to a post-composition by a Möbius transformation  $A \in \text{Aut}(\hat{\mathbb{C}})$ .*

*Moreover, the solutions vary analytically in the parameter  $\mu \in L^\infty(\hat{\mathbb{C}})$ , namely, for any fixed  $z \in \hat{\mathbb{C}}$ , the assignment*

$$z \mapsto f^\mu(z)$$

*is analytic in  $\mu$ .*

### 3. THE BERS EMBEDDING

**3.1. Construction of the embedding:** As earlier, we fix a Riemann surface  $X = \mathbb{H}/\Gamma$ . In this section we describe the *Bers embedding*

$$(2) \quad \mathcal{B}_X : \mathcal{T}_{g,n} \rightarrow \mathbb{C}^{3g-3+n}.$$

We refer to the image as the *Bers domain*. The fact that the complex structure on  $\mathcal{T}_{g,n}$  acquired via this embedding does not depend on the choice of basepoint  $X$  is due to the analytical dependence of parameters in Theorem 2.1.

The starting point is the Simultaneous Uniformization Theorem of Bers ([7], see Theorem 3.3) which associates to a point  $(Y, g) \in \mathcal{T}(X)$  a quasiconformal map

$$F : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

that fixes  $\{0, 1, \infty\}$ , and conjugates the action of  $\Gamma$  to the action of a *Kleinian* group  $\Gamma_Y < \text{Aut}(\hat{\mathbb{C}}) = \text{PSL}_2(\mathbb{C})$ , such that

- $F$  is conformal on  $\mathbb{H}_-$ , where  $\mathbb{H}_\pm$  denotes the upper/lower half-plane respectively, and
- $\Gamma_Y$  leaves invariant the images of  $\mathbb{H}_-$  and  $\mathbb{H}_+$  with quotients  $\overline{X}$  and  $Y$  respectively.

The idea of the proof of existence of such a quasiconformal map  $F$  is simple: lift the Beltrami coefficient  $\mu(g)$  to the upper half plane  $\mathbb{H}_+$ , and extend to a Beltrami coefficient  $\hat{\mu}$  on  $\hat{\mathbb{C}}$  by defining it to be identically zero on  $\mathbb{H}_-$ . The map  $F$  above is then the normalized solution to the Beltrami equation with the Beltrami coefficient  $\hat{\mu}$ . Since  $\hat{\mu}$  is  $\Gamma$ -invariant, we have, by the uniqueness of solutions (see Theorem 2.1) that

$$(3) \quad F \circ \gamma = \rho(\gamma) \circ F \text{ for each } \gamma \in \Gamma_Y$$

where  $\rho : \pi_1(X) \rightarrow \text{PSL}_2(\mathbb{C})$  has an image  $\Gamma_Y$ , a Kleinian group.

Note that  $F|_{\mathbb{H}_-}$  is conformal as  $\hat{\mu}|_{\mathbb{H}_-} \equiv 0$ , and is moreover univalent as it is the restriction of a homeomorphism.

Recall that the *Schwarzian derivative* of a univalent function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is the meromorphic quadratic differential

$$(4) \quad S(f)(z) = \left( \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right) dz^2$$

that measures the deviation of  $f$  from being Möbius. In particular,  $S(f)$  vanishes identically if and only if  $f \in \text{Aut}(\hat{\mathbb{C}})$ .

By the equivariance (3), and the property that

$$S(A \circ f) = S(f) \text{ for any } A \in \text{Aut}(\hat{\mathbb{C}}),$$

the Schwarzian derivative of  $F|_{\mathbb{H}_-}$  thus defines a  $\Gamma_Y$ -invariant quadratic differential that descends to a meromorphic quadratic differential  $\phi_Y$  on the Riemann surface  $\bar{X}$ . The Schwarzian derivative has poles of order at most one, and by Riemann–Roch, we know that the space of such quadratic differentials  $\hat{\mathcal{Q}}(X) \cong \mathbb{C}^{3g-3+n}$ . The assignment

$$\mathcal{B}_X(Y) = \phi_Y$$

defines the map in (2).

The fact that this is well-defined relies on the observation that if  $g$  and  $h$  are homotopic quasiconformal maps, then the corresponding maps  $F_1|_{\mathbb{H}_+}$  and  $F_2|_{\mathbb{H}_+}$  determine the same maps on  $\mathbb{R}$  that is the common boundary of both half-planes  $\mathbb{H}_-$  and  $\mathbb{H}_+$ . The conformal maps  $F_1|_{\mathbb{H}_-}$  and  $F_2|_{\mathbb{H}_-}$  consequently determine the same Schwarzian derivatives.

A similar argument proves that  $\mathcal{B}_X$  is an embedding: if the Schwarzian derivatives  $\phi_{Y_1}$  and  $\phi_{Y_2}$  agree on  $\mathbb{H}_-$ , then  $F_1|_{\mathbb{H}_-}$  and  $F_2|_{\mathbb{H}_-}$  must differ by an automorphism of  $\mathbb{H}_-$  that fixes  $\{0, 1, \infty\}$  on the boundary  $\mathbb{R}$ ; consequently  $F_1|_{\mathbb{R}} = F_2|_{\mathbb{R}}$  and considering the restrictions  $F_1|_{\mathbb{H}_+}$  and  $F_2|_{\mathbb{H}_+}$ , we see that the surfaces  $Y_1$  and  $Y_2$  are equivalent in  $\mathcal{T}_{g,n}$ .

The Bers embedding we just described has the following two key properties, that we shall now briefly discuss:

**$\mathcal{B}_X$  is bounded:** We equip the vector space  $\mathcal{Q}(\mathbb{H})$  of holomorphic quadratic differentials on  $\mathbb{H}$  with the norm

$$\|q(z)dz^2\| = \sup_{z \in \mathbb{H}} 4\Im(z)^2 |q(z)|.$$

The invariance of the 1-form  $\frac{dz}{\Im(z)}$  under the Möbius group implies that the norm is invariant under the Fuchsian group  $\Gamma$  and determines a finite norm on  $\hat{\mathcal{Q}}(X)$ . With respect to this norm, we have:

**Theorem 3.1** (Nehari). *The image of  $\mathcal{B}_X$  is contained in the ball  $B(0, 3/2) \subset \widehat{\mathcal{Q}}(X)$ .*

The proof of this is a short computation involving (4) and an application of the classical Area Theorem; the reader is referred to [19] or [3] for details.

$\mathcal{B}_X$  is open: For a quadratic differential  $q(z)dz^2 \in \mathcal{Q}(\mathbb{H})$ , define the Beltrami coefficient  $\mu_q$  on  $\mathbb{C}$  by

$$(5) \quad \mu_q(z) = 2y^2 q(\bar{z})$$

on  $\mathbb{H}_+$  and identically zero on  $\mathbb{H}_-$ .

Let  $f^{\mu_q}$  be the solution of the Beltrami equation as in Theorem 2.1. We then have:

**Theorem 3.2** (Ahlfors–Weill [4]). *If  $\|q\| < 1/2$ , the Schwarzian derivative of the restriction of  $f^{\mu_q}$  to the upper half-plane is  $q$ .*

The idea is to consider the two linearly independent solutions  $\eta_1, \eta_2$  of the Schwarzian equation:

$$(6) \quad \eta'' + \frac{1}{2}q\eta = 0$$

on the lower half-plane, whose ratio  $f = \eta_1/\eta_2$  has Schwarzian derivative  $q$ .

It can be checked that  $f$  is injective, and the map

$$F(z) = \frac{\eta_1(z) + (\bar{z} - z)\eta'_1(z)}{\eta_2(z) + (\bar{z} - z)\eta'_2(z)}$$

defines an extension to the upper half-plane which is quasiconformal (this uses the bound on the norm of  $q$ ) having Beltrami coefficient (5). See Chapter VI.C of [3] for details.

An equivariant version of this then defines an inverse of the Bers embedding  $\mathcal{B}_X$  in an open ball  $B(0, 1/2) \subset \widehat{\mathcal{Q}}(X)$ . This construction works for (smaller) neighborhoods of an arbitrary point in the image, showing that the image of the embedding is open.

**3.2. Structure of the Bers boundary:** In what follows, a *Bers domain* shall be the image of any Bers embedding  $\mathcal{B}_X$ ; the geometry of this domain in complex space can be studied by analyzing the construction sketched in the previous section.

First, the images of the upper and lower half-planes  $\mathbb{H}_\pm$  by the quasiconformal map  $F$  in (3) are quasidisks. The image group  $\Gamma_Y < \mathrm{PSL}_2(\mathbb{C})$  of the representation  $\rho$  leaves these invariant, and the quotient of each yields the Riemann surfaces  $\overline{X}$  and  $Y$  respectively. Such a representation is called *quasi-Fuchsian* and thus Bers proved what is known as the *Simultaneous Uniformization Theorem*:

**Theorem 3.3** (Bers, [7]). *The space of quasi-Fuchsian representations  $\mathcal{QF}(\pi_1(X))$  is homeomorphic to  $\mathcal{T}_{g,n} \times \mathcal{T}_{g,n}$ .*

The Bers embedding  $\mathcal{B}_X$  as in (2) is thus a slice of the subset  $\mathcal{QF}(\pi_1(X))$  of the  $\mathrm{PSL}_2(\mathbb{C})$  representation variety (when the conformal structure on one of the factors is kept fixed at  $X$ ).

The image of a quasi-Fuchsian representation  $\Gamma_Y \in \mathcal{QF}(\pi_1(X))$  is a *Kleinian group*, a discrete subgroup of  $\mathrm{PSL}_2(\mathbb{C})$  that acts freely and properly-discontinuously on  $\mathbb{H}^3$  with quotient a hyperbolic 3-manifold  $M$ . Topologically,  $M$  is the interior of the product  $S \times I$  of a surface and an interval, where  $S$  is the underlying topological surface of  $X$ , and the boundary components  $S \times \{0, 1\}$  acquire a conformal structure from the action of  $\Gamma$  at the sphere at infinity  $\partial\mathbb{H}^3$ .

It is well-known that discrete faithful representations form a *closed* subset, denoted by  $\mathrm{AH}(S)$ , of the  $\mathrm{PSL}_2(\mathbb{C})$  representation variety; in particular, the representations that arise in the boundary of a Bers slice correspond to hyperbolic 3-manifolds with cusps, when the conformal structure on either boundary degenerates by pinching a curve, or degenerate “ends”, when the conformal structure degenerates by pinching an “ending lamination” (see [35] for a survey).

Thus points in the Bers boundary can be studied using tools from hyperbolic geometry; see §3.2 for some results for the case when  $S$  is a punctured torus. As an example, Sullivan’s theorem on rigidity of the “totally degenerate” hyperbolic 3-manifolds yield the fact that such boundary points are *peak convex*, that is,

**Theorem 3.4** (Theorem 2 in [45]). *Let  $f : \Delta \rightarrow \partial\mathcal{B}_X$  be a holomorphic map such that  $f(0)$  is a totally degenerate boundary point. Then  $f$  is constant.*

In contrast, at any non-maximal cusp, the remaining moduli of the conformal boundary parametrize an open set in  $\partial\mathcal{B}_X$ .

The simplest Teichmüller spaces are those of the punctured torus and the four-punctured sphere, as they have complex dimension 1. Much is known about the Bers embedding in these cases. See [25], [24] for visualizations, computational aspects, which rely on solving the Schwarzian equation numerically and then checking for discreteness of the resulting holonomy.

We highlight here some results for the case when  $X$  is a punctured torus:

- The Bers domain  $\mathcal{B}_X$  is a Jordan domain; this is a consequence of the deep work of Y. Minsky in [35] that proves the Ending Lamination Conjecture for punctured-torus groups. The boundary circle can be identified with projective classes of measured laminations on the punctured torus.
- H. Miyachi proved a folklore conjecture in [36] showing that the rational points on the Bers boundary of  $\mathcal{B}_X$  are  $2/3$  cusps; in particular, though a Jordan domain, it is not a quasi-disk.
- Boundary points exhibit spiralling behavior, which is described in work of D. Goodman in [17].

The last two papers crucially use Minsky's Pivot Theorem in [35] and study the behaviour of *rational pleating rays* introduced by L. Keen and C. Series in [21].

The following question is still open, however:

**Question 3.5** (Canary in [11]). *What is the Hausdorff dimension of the boundary of  $\mathcal{B}_X$  when  $X$  is a once-punctured torus?*

For Bers domains of higher dimensions, very little is known. In particular, it is not known whether the closure is homeomorphic to a closed ball. In fact, the following is still open:

**Conjecture 1** (Bromberg in [10]). *The closure of a Bers domains of dimension greater than one in  $\mathcal{QF}(S)$  is not locally connected.*

One can also raise the following:

**Question 3.6.** *What does the tiling of the Bers domain by fundamental domains for the mapping class group action look like? In particular, for a fixed fundamental domain  $F$ , does the Euclidean diameter of  $\gamma_i \cdot F$  tend to zero for a diverging sequence  $\gamma_i \rightarrow \infty$  (for  $i \rightarrow \infty$ ) of mapping classes in  $\text{MCG}(S)$ ?*

#### 4. INVARIANT METRICS ON TEICHMÜLLER DOMAINS

**4.1. The Kobayashi and Carathéodory metrics.** The Bers domain is often studied in terms of metrics invariant under the automorphism group of the domain. We will focus on the Kobayashi and Carathéodory metrics, which are complete Finsler metrics intimately related to issues of Euclidean convexity of the domain.

On any bounded domain  $\Omega$  in  $\mathbb{C}^N$ , the *infinitesimal Kobayashi metric* is defined by the following norm for a tangent vector  $v$  at a point  $X \in \Omega$ :

$$(7) \quad K_\Omega(X, v) = \inf_{h: \Delta \rightarrow \Omega} \frac{|v|}{|h'(0)|}$$

where the infimum is over all holomorphic maps  $h: \Delta \rightarrow \Omega$  such that  $h(0) = X$  and  $h'(0)$  is a multiple of  $v$ .

The *Kobayashi metric*  $d_\Omega^K$  on  $\Omega$  is then the distance defined in the usual way: one first defines lengths of piecewise  $C^1$  curves in  $\Omega$  using the above norm and then takes the infimum of lengths of curves joining two given points to get the distance between them.

The *Carathéodory metric*  $d_\Omega^C$  on  $\Omega$  is defined by

$$d_\Omega^C(p, q) = \sup\{d_\mathbb{H}(f(p), f(q))\},$$

where  $d_\mathbb{H}$  denotes the Poincaré metric on  $\Delta$  and the supremum is over all holomorphic maps from  $\Omega$  to  $\Delta$ . Note that  $d_\mathbb{H}$  is the distance function associated to the Kähler metric  $4 \frac{dz \otimes \bar{dz}}{(1-|z|^2)^2}$  on  $\Delta$ .



The following is immediate from the definitions: For any  $p, q \in \Omega$  we have

$$(8) \quad d_{\Omega}^C(p, q) \leq d_{\Omega}^K(p, q)$$

and we have the contracting property

$$(9) \quad d_{\Omega_2}^K(f(p), f(q)) \leq d_{\Omega_1}^K(p, q) \text{ and } d_{\Omega_2}^C(f(p), f(q)) \leq d_{\Omega_1}^C(p, q)$$

for any holomorphic map  $f : \Omega_1 \rightarrow \Omega_2$  between the domains.

*Remark.* The Kobayashi and the Carathéodory metrics coincide on the unit ball  $\mathbb{B}^n \subset \mathbb{C}^n$ . Moreover, they are equal to the norm of the complex hyperbolic metric on  $\mathbb{B}^n$ . In particular,  $d_{\Delta}^K$  is the Poincaré (hyperbolic) metric on the ball.

**4.2. Complex Geodesics.** Let  $\Omega \subset \mathbb{C}^n$  be a bounded domain. A *complex geodesic* in  $\Omega$  is a holomorphic and isometric embedding  $\tau : (\mathbb{H}, d_{\mathbb{H}}^K) \rightarrow (\Omega, d_{\Omega}^K)$ . As noted above, the Kobayashi  $d_{\mathbb{H}}^K$  on  $\mathbb{H}$  is the distance function of the hyperbolic metric, and is isometric to the Poincaré disk  $(\Delta, d_{\mathbb{H}^2})$ .

It is a remarkable fact, due to Lempert [29] and generalized by Royden–Wong [43], that complex geodesics exist in abundance when  $\Omega$  is a bounded convex domain:

**Theorem 4.1** (Lempert[29], Royden–Wong[43]). *Let  $\Omega \subset \mathbb{C}^n$  be a bounded convex domain.*

(i) *For any  $p, q \in \Omega$  there exists a complex geodesic  $\tau : \mathbb{H} \rightarrow \Omega$  with  $\tau(0) = p$  and  $\tau(z) = q$  for some  $z \in \mathbb{H}$ .*

(ii) *For any  $p \in \Omega$  and  $v \in \mathbb{C}^n$  there exists a complex geodesic  $\tau : \mathbb{H} \rightarrow \Omega$  with  $\tau(0) = p$  and  $\tau'(0) = tv$  for some  $t > 0$ .*

This was proved for domains with  $\mathcal{C}^2$ -smooth strongly convex domains by Lempert and generalized to arbitrary convex domains by Royden and Wong.

Apart from convex domains there are very few domains for which the existence of complex geodesics with prescribed data is known. Interestingly, this is known to be true for Teichmüller domains:

**Theorem 4.2** (Earle–Kra–Krushkal [12]). *Let  $\Omega \subset \mathbb{C}^n$  be a Teichmüller domain. Then the conclusions of Theorem 4.1 hold.*

We will describe complex geodesics in Teichmüller space in more detail in the next subsection.

Returning to convex domains, more is known about complex geodesics, apart from their existence. In fact, Lempert and Royden–Wong–Krantz proved the following:

**Theorem 4.3** (Lempert[29], Royden–Wong–Krantz[43]). *Let  $\Omega \subset \mathbb{C}^n$  be a bounded convex domain. For every complex geodesic  $\tau : \mathbb{H} \rightarrow \Omega$  there is a holomorphic*

retract  $\Phi : \Omega \rightarrow \Omega$  for  $\tau$ , i.e., there is a holomorphic map  $\Phi : \Omega \rightarrow \Omega$  such that  $\Phi(\Omega) \subset \tau(\mathbb{H})$  and  $\Phi(z) = z$  for all  $z \in \tau(\mathbb{H})$ .

By composing  $\Phi$  with  $\tau^{-1}$  we get a map, which we continue to denote by  $\Phi$ , from  $\Omega$  to  $\mathbb{H}$ . It is not difficult to see that  $\Phi$  is an extremal map for the Carathéodory metric and, in fact, we have

**Corollary 4.4.** *If  $\Omega \subset \mathbb{C}^n$  is a bounded convex domain, then  $d_\Omega^K = d_\Omega^C$ .*

Theorems 4.1 and 4.2 prompt a natural question: *can a Teichmüller domain be bounded and convex?* On the other hand, an analogue of Theorem 4.3 and Corollary 4.4 was not known to hold for Teichmüller space. In fact, Markovic proved recently that Corollary 4.4 fails for Teichmüller space, thereby answering the question in the negative. We will discuss his work and related issues in later sections.

**4.3. The Teichmüller metric.** It turns out that the Kobayashi metric on  $\mathcal{T}_{g,n}$  can be described in terms of extremal quasiconformal maps. In fact, the following definition of the *Teichmüller metric* predates that of the Kobayashi metric.

The *Teichmüller distance* between two marked surfaces  $X$  and  $Y$  is defined by

$$(10) \quad d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_f \ln K(f)$$

where the infimum is taken over quasiconformal homeomorphisms preserving the marking and fixing the punctures, and

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

where  $\mu$ , as in (1), is the *quasiconformal dilatation* of  $f$ .

The infimum is, in fact, attained by a *Teichmüller map*  $\Psi : X \rightarrow Y$ . It turns out that such a map is an affine stretch in local Euclidean charts on  $X$  associated to a holomorphic quadratic differential  $q$  on  $X$ . Indeed, if the quadratic differential  $q$  can be expressed as  $d\xi^2$  in a local chart given by the complex coordinate  $\xi = x + iy$ , then a Teichmüller map can be written as

$$\Psi(\xi) = K^{1/2}x + iK^{-1/2}y,$$

where  $K$  is the minimum dilation in (10).

A remarkable feature of the Teichmüller metric is that it admits a geometric description of geodesic paths: namely, the one real-parameter worth of such affine stretches

$$(11) \quad \Psi_t(\xi) = e^t x + i e^{-t} y$$

yields a family of surfaces  $Y_t$  that lie along a (parametrized) Teichmüller geodesic ray. These, in fact, extend to complex geodesics, as defined earlier, by allowing a complex parameter  $\tau = t + i\theta \in \mathbb{H}$ , in which case

$$(12) \quad \Psi_\tau(\xi) = e^t \cos \theta x + i e^{-t} \sin \theta y$$

in each coordinate chart. The locus of Riemann surfaces obtained by varying  $\tau \in \mathbb{H}$  is then a totally geodesic disk in Teichmüller space called a *Teichmüller disk*. The induced metric on the disk is then the Poincaré (hyperbolic) metric on  $\mathbb{H}$ .

Alternatively, a holomorphic quadratic differential  $q \in \mathcal{Q}(X)$  gives rise to a holomorphic embedding  $\mathcal{D}(q) : \mathbb{H} \rightarrow \mathcal{BD}(X)$  defined by

$$\mathcal{D}_q(\tau) = \frac{1 - \tau |q|}{1 + \tau q}.$$

On the other hand, as in §2, we have a natural holomorphic map  $\pi : \mathcal{BD}(X) \rightarrow \mathcal{T}_{g,n}$  which associates to  $\mu$  the quasiconformal map  $[\mu]$  with complex dilatation  $\mu$ . It can be checked that the composition  $\pi \circ \mathcal{D}(q) : \mathbb{H} \rightarrow \mathcal{T}_{g,n}$  coincides with the Teichmüller disk above. Moreover, such a disk is a complex geodesic as defined in the previous section (§4.2) (See for example, §9.5 of [28], for more on their construction and properties). As a side note, we point out a recent result of S. Antonakoudis [5] that any such isometric embedding of  $(\mathbb{D}, d_\rho)$  in  $(\mathcal{T}_{g,n}(X), d_{\mathcal{T}})$  is necessarily holomorphic or anti-holomorphic.

By the work of Royden, we have

**Theorem 4.5** (Royden [42]). *The Kobayashi and Teichmüller metrics coincide.*

*Remark.* The above construction in fact gives a Teichmüller disk  $\mathcal{D}$  through any point and any (complex) direction; using the fact that it is an isometric embedding, the contracting property of the Kobayashi metric (see (9)) implies that  $d^K \leq d_{\mathcal{T}}$ ; the main work is to prove the reverse inequality, see [12] or Chapter 7 of [16] for a proof using the Slodkowski extension theorem.

In what follows, let  $S$  be a closed orientable surface of genus  $g \geq 2$ ; the discussion extends to the case of a hyperbolic surface with punctures which we elide. The mapping class group  $\text{MCG}(S)$  is the group of self-homeomorphisms of  $S$  up to homotopy. It is immediate from the definitions that each element of  $\text{MCG}(S)$  is a holomorphic isometry of  $(\mathcal{T}_g, d_{\mathcal{T}})$ . It is a fact, due to Royden again, that these are essentially all the isometries:

**Theorem 4.6** (Royden [42]).  *$\text{Isom}(\mathcal{T}_g, d_{\mathcal{T}}) = \text{MCG}(S)$  when  $g \geq 3$ . For  $g = 2$ , the homomorphism  $\text{MCG}(S)$  to  $\text{Isom}(\mathcal{T}_g, d_{\mathcal{T}})$  has a kernel of order two, generated by the hyperelliptic involution (that acts trivially on  $\mathcal{T}_g$ ).*

Since biholomorphic mappings are isometries for the Kobayashi metric, this implies

**Corollary 4.7.**  *$\text{Aut}(\mathcal{T}_g)$  equals  $\text{MCG}(S)$  when  $g \geq 3$ , and when  $g = 2$ , equals its quotient by the order-two subgroup generated by the hyperelliptic involution.*

Note that by (12), Teichmüller disks arise as orbits of the linear group  $\text{SL}_2(\mathbb{R})$  which act linearly on the local  $\xi = x + iy$  coordinates. A full account of the dynamics of this action is beyond the scope of this article (see [53] for a readable survey), however we mention the following fundamental ergodicity result which we will use later in the paper. We note first that the identification of the cotangent space  $T_X^* \mathcal{T}_g$  with the vector space  $\mathcal{Q}(X)$  of holomorphic quadratic differentials on  $X$ , combined with (11) gives rise to the *Teichmüller geodesic flow*  $\Psi_t$  on the unit cotangent bundle  $T_1^* \mathcal{T}_g$ .

**Theorem 4.8** (Masur [32], Veech [50]). *There is a  $\text{MCG}(S)$ -invariant measure on the unit cotangent bundle  $T_1^* \mathcal{T}_g$  such that*

- (i) *the Teichmüller geodesic flow  $\phi_t$  on  $T_1^* \mathcal{T}_g$  is ergodic, and*
- (ii) *the induced measure on  $\mathcal{T}_g$  is equivalent to the top-dimensional Hausdorff measure of  $d_{\mathcal{T}}$ .*

We end this section by remarking that there are other invariant metrics on Teichmüller which are important in other settings and have been extensively studied, notably the Weil-Petersson metric, which is an incomplete Kähler metric on  $\mathcal{T}_{g,n}$  (for a survey see [52]). For comparisons between various metrics, see [55] or the series of papers of Liu–Sun–Yau (for example [30]).

## 5. COMPARISON WITH PRODUCTS AND BOUNDED SYMMETRIC DOMAINS

It is known that Teichmüller space endowed with the Teichmüller metric  $d_{\mathcal{T}}$  is not negatively curved, even in a “coarse” sense. More precisely, it is not Gromov hyperbolic (see [33]). On the other hand, quite remarkably,  $\mathcal{T}_{g,n}$  exhibits some features of complex structures of strictly negatively curved Kähler manifolds. In particular, one has the following “rigidity” result :

**Theorem 5.1** (Corollary 1 of [48]). *Let  $f : \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{T}_{g,n}$  be a holomorphic map such that  $f|_{\mathbb{D} \times \{w\}} : \mathbb{D} \rightarrow \mathcal{T}_{g,n}$  is proper for some  $w \in \mathbb{D}$ . Then  $f(z, w_1) = f(z, w_2)$  for all  $z, w_1, w_2 \in \mathbb{D}$ .*

An immediate corollary is

**Corollary 5.2** ([48], [37]).  *$\mathcal{T}_{g,n}$  is not a product of complex manifolds.*

The proof of Theorem 5.1 uses the interpretation of the Bers domain as a subset of quasi-Fuchsian characters; here we provide a sketch of the proof:

Recall from §3 that the “totally degenerate” points on the Bers boundary are of full measure; this remains true for the boundary limits of any holomorphic disk as in (10) with respect to the Lebesgue measure on the boundary circle (such limits exist in almost every radial direction by Fatou’s theorem). However, as a

consequence of Sullivan rigidity, totally degenerate boundary points do not admit holomorphic deformations (*cf.* Theorem 3.4). The proof of the rigidity result can then be derived from the abundance of such rigid boundary points, and an application of F. and M. Riesz's theorem that says that if a holomorphic map on the disk extends to a map that is constant almost everywhere on the boundary, then it is constant.

When  $g = 1$ , the Teichmüller space  $\mathcal{T}_1$  is biholomorphic to the upper half-plane  $\mathbb{H}^2$ , so its geometry is that of this symmetric space. Indeed, the factor  $\frac{1}{2}$  in the definition of the Teichmüller metric in (10) is to ensure that it coincides with the hyperbolic metric.

When  $g \geq 2$ , the discreteness of the automorphism group (Corollary 4.7) immediately implies that  $\mathcal{T}_g$  is never biholomorphic to a bounded symmetric domain. Even stronger still is the result of Farb–Weinberger who showed that the isometry group of *any* MCG( $S$ )-invariant complete finite co-volume Finsler metric on  $\mathcal{T}_g$  has to be a finite extension of MCG( $S$ ) if  $g \geq 3$  (see [13]). In this connection we also note the result of S. Antonakoudis who proved in [6] that, except for the torus case, there are no holomorphic isometric embeddings of Teichmüller space into a bounded symmetric domain (or vice versa).

However, the analogy with symmetric spaces continues to shed light on the geometry of Teichmüller space and its compactifications (see, for example, [20] for an account of the latter).

## 6. CONVEXITY AND PSEUDOCONVEXITY

**Pseudoconvexity of Teichmüller domains:** The notion of “pseudoconvexity” of a domain (or more generally, a complex manifold) is fundamental in complex geometry, as it is closely related to the notion of a “domain of holomorphy” or a “Stein manifold” (see, for example, the insightful survey of Siu in [46]). We provide a brief discussion here.

A domain  $\Omega \subset \mathbb{C}^n$  is said to be *strictly (or strongly) pseudoconvex* if there is a  $\mathcal{C}^2$ -smooth function  $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$  such that

- (1)  $\Omega = \{z \in \mathbb{C}^n : \varphi(z) < 0\}$

- (2)  $\varphi^{-1}(0) = \partial\Omega$  and  $\nabla\varphi \neq 0$  on  $\partial\Omega$

- (3) the following convexity property is satisfied at any  $p \in \partial\Omega$ : the Hermitian quadratic form (the *Levi form*) defined by

$$\mathcal{L}_\varphi(p; v) = \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p) v_i \bar{v}_j$$

is positive for all  $v \in \mathbb{C}^n \setminus \{0\}$  such that  $v_N := \langle \nabla_{\bar{z}} \varphi(p), v \rangle = 0$ . (Here  $\nabla_{\bar{z}} \varphi(p) = (\frac{\partial \varphi}{\partial \bar{z}}, \dots, \frac{\partial \varphi}{\partial \bar{z}})$ .)

Note that the condition  $v_N = 0$  is equivalent to the requirement that  $v$  lies in the maximal complex subspace of  $T_p(\partial\Omega)$ .

It can be shown that the above condition can be restated as follows:  $D$  is strictly pseudoconvex at  $p \in \partial D$  if there exists a neighborhood  $U$  of  $p$  in  $\mathbb{C}^n$  such that  $\partial D \cap U$  is  $\mathcal{C}^2$ -smooth and a biholomorphism  $\Phi : U \rightarrow \Phi(U) \subset \mathbb{C}^n$  such that  $\Phi(U \cap \bar{D})$  is strictly convex at  $p$ , that is, there is a local defining function for the boundary defined near  $p$  that has a positive-definite Hessian.

More generally, a bounded domain  $D \subset \mathbb{C}^n$  is *pseudoconvex* if it can be written as a union  $D = \bigcup_{i=1}^{\infty} D_i$  where each  $D_i$  is strictly pseudoconvex and  $\bar{D}_i \subset D_{i+1}$ .

Note that we do not require  $\partial D$  to be  $\mathcal{C}^2$ -smooth to define pseudoconvexity. It is clear that pseudoconvexity is invariant under biholomorphisms between domains. Examples of pseudoconvex domains include convex domains: this can be seen, for instance, by constructing a  $\mathcal{C}^\infty$ -smooth *strongly convex* exhaustion function on such a domain (see Theorem 2.40 of [27]).

The pseudoconvexity of Bers domains was established by Bers and Ehrenpreis in [9] (see also [44] for the stronger property of “polynomial convexity”). In [51], Wolpert proved that the geodesic length function of a “filling” system of curves (*i.e.* one whose complement is a disk) is a smooth proper, strictly plurisubharmonic exhaustion function on  $\mathcal{T}_{g,n}$  (*plurisubharmonic* means that the function has a positive definite Levi form as in condition (3) at every point).

Another such exhaustion was provided by A. J. Tromba in [49], where he showed that the Dirichlet energy of harmonic diffeomorphism  $h : Y \rightarrow X$  between hyperbolic surfaces, as  $Y$  varies in  $\mathcal{T}_{g,n}$ , is strictly plurisubharmonic.

In [54] S.-K. Yeung constructed *bounded* smooth strictly plurisubharmonic exhaustion functions. In [38] Miyachi constructed other plurisubharmonic exhaustions by considering extremal lengths of filling pairs of measured foliations.

**Convexity of Teichmüller domains.** Unlike pseudoconvexity, Euclidean convexity of domains is not natural in the setting of complex geometry, as it is not preserved under biholomorphisms. Nevertheless, there are several important results about the complex geometry of such domains. In particular, one has the striking results of Lempert, which we touched upon in §4.2.

A basic question regarding the geometry of Teichmüller domains is the following folklore conjecture, recently proved by Markovic:

**Conjecture 2.** *The Teichmüller space of a closed surface of genus  $g \geq 2$  is not biholomorphic to a bounded convex domain  $\Omega \subset \mathbb{C}^{3g-3}$ .*

As described in S.-T. Siu’s survey [47], Conjecture 2 fits in the context of the following theorem of S. Frankel:

**Theorem 6.1** (Frankel [15]). *Let  $\Omega \subset \mathbb{C}^n$  a bounded convex domain. If there is a discrete subgroup  $\Gamma \subset \text{Aut}(\Omega)$  acting freely on  $\Omega$  such that  $\Omega/\Gamma$  is compact then  $\Omega$  is biholomorphic to a bounded symmetric domain.*

As for  $\mathcal{T}_g$ , it is known that there is a discrete subgroup  $\Gamma \subset \text{Aut}(\mathcal{T}_g)$  acting freely on  $\mathcal{T}_g$  such that the quotient  $\mathcal{T}_g/\Gamma$  has finite Kobayashi volume. However  $\mathcal{T}_g/\Gamma$  is noncompact and a generalization of Frankel's result to the case of convex domains with finite-volume quotients is not known.

By Corollary 4.4, the existence of a convex embedding of  $\mathcal{T}_g$  would imply that the Kobayashi and Carathéodory metrics coincide, that is, we have  $d_{\mathcal{T}}^K = d_{\mathcal{T}}^C$ . In fact, it was proved by I. Kra [26] that these two metrics agree on Teichmüller disks associated to Abelian differentials. The recent result of Markovic alluded to above is the stronger fact that

**Theorem 6.2** (Markovic [31]). *If  $g \geq 2$ , we have  $d_{\mathcal{T}}^K \neq d_{\mathcal{T}}^C$  on  $\mathcal{T}_g$ .*

In particular,  $\mathcal{T}_g$  cannot be biholomorphic to a bounded convex domain, answering Conjecture 2.

Around the same time, the authors of the current article proved the following *local* property of the boundary of a Teichmüller domain.

We say that a domain  $\Omega \subset \mathbb{C}^n$  is *locally convex* at  $p \in \partial\Omega$  if  $\Omega \cap B(p, r)$  is convex for some  $r > 0$ , where  $B(p, r)$  denotes the Euclidean ball with center  $p$  and radius  $r$ . Moreover, it is locally *strictly convex* if  $\Omega \cap B(p, r)$  is strictly convex.

**Theorem 6.3** (Gupta–Seshadri [18]). *For  $g \geq 2$ , any Teichmüller domain for  $\mathcal{T}_g$  cannot have any locally strictly convex point on its boundary.*

In fact, minor modifications of the proof yield a more general statement: a Teichmüller domain cannot have a locally strictly *convexifiable* boundary point. By definition, if  $p \in \partial\Omega$  is such a point, then there exists  $r > 0$  and a holomorphic embedding  $F : \Omega \cap B(p, r) \rightarrow \mathbb{C}^{3g-3}$  such that  $F(\Omega \cap B(p, r))$  is strictly convex.

We remark that this local convexity is significantly weaker than that of global convexity of the domain; indeed, it is an elementary and well-known fact that any bounded domain with  $C^2$ -smooth boundary has a locally strictly convex boundary point (see §4.1 of [18]). If we consider local *convexifiability*, we have a more concrete example: it is shown in [40] that the domain

$$D_\epsilon = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \sqrt{|z_1 - \bar{z}_1 z_2|^2 + \epsilon} + |z_2|^2 < 1 \right\}$$

is, for small enough  $\epsilon$ , strongly pseudoconvex, diffeomorphic to  $\mathbb{B}^2$  and not biholomorphic to a convex domain. The strong pseudoconvexity property implies that every boundary point is locally strictly convexifiable in the sense above.

One motivation for Theorem 6.3 is the technique of *localization* in complex analysis, that we shall elaborate on in §8. A version of that was used by Frankel

in his proof of Theorem 6.1, and he observes in the paper that one can replace the assumption of global convexity of the domain with a suitable local convexity at certain orbit accumulation points.

The next two sections are devoted to discussions of the above results of Markovic and Gupta-Seshadri.

## 7. KOBAYASHI $\neq$ CARATHÉODORY ON TEICHMÜLLER DOMAINS

This section provides an extended discussion of some of the ideas involved in Markovic's proof of Theorem 6.2 proving that the Carathéodory and Kobayashi metrics are not equal on Teichmüller domains.

His first observation is that  $\mathcal{T}_{0,5}$ , the Teichmüller space of the five-punctured sphere, admits a holomorphic embedding into  $\mathcal{T}_g$  for any  $g \geq 2$  that is also an isometric embedding with respect to the Teichmüller metrics.

Hence, by the distance-decreasing properties of the two metrics and the domination of the Carathéodory metric by the Kobayashi metric (see (9) and (8) in §4), it suffices to prove that  $d_{\mathcal{T}_{0,5}}^K \neq d_{\mathcal{T}_{0,5}}^C$ .

This involves several steps, and crucially relies on the following observation regarding the restrictions of these metrics to Teichmüller disks:

Let  $q \in \mathcal{Q}(X)$  be a holomorphic quadratic differential and  $\mathcal{D}_q$  the Teichmüller disk associated to  $q$  as in (12). We say that  $\mathcal{D}_q$  is *extremal* for a holomorphic function  $\Phi : \mathcal{T}_{g,n} \rightarrow \mathbb{H}$  if  $\Phi \circ \mathcal{D}_q \in \text{Aut}(\mathbb{H})$ .

Properties (8) and (9) of the Kobayashi and Carathéodory metrics then imply that the two metrics agree on  $\mathcal{D}_q$  if and only if  $\mathcal{D}_q$  is extremal for some holomorphic map  $\Phi : \mathcal{T}_{g,n} \rightarrow \mathbb{H}$ .

### Step 1 : A criterion for $d_{\mathcal{T}}^K = d_{\mathcal{T}}^C$ on certain Teichmüller disks in $\mathcal{T}_{0,5}$ .

We say that  $q \in \mathcal{Q}(X)$  is a *Jenkins-Strebel* differential (*J-S* differential, in short) if  $q$  induces a decomposition of  $X$  into a finite number of annuli  $\Pi_j$ ,  $j = 1, \dots, k$ , foliated by closed horizontal trajectories of  $q$ . (Recall that a horizontal trajectory is an integral curve of directions  $v$  where the quadratic form  $q(v, v) > 0$ .) Let  $m_j$  denote the conformal modulus of  $\Pi_j$ . If the  $m_j$  have rational ratios then  $q$  is said to be a *rational* J-S differential.

Assume that  $q$  is rational. Let  $\gamma_1, \dots, \gamma_k$  be a collection of disjoint simple closed curves on  $X$ , with  $\gamma_j$  homotopic to  $\Pi_j$ . Markovic observes that the Teichmüller disk  $\mathcal{D}_q$  determined by  $q$  is stabilized by the infinite cyclic subgroup of  $\text{MCG}(S)$  generated by a product  $T$  of certain powers of Dehn twists about  $\gamma_j$ .

An averaging procedure then yields the following statement: if  $\mathcal{D}_q$  is extremal for some holomorphic function  $\hat{\Phi} : \mathcal{T}_{g,n} \rightarrow \mathbb{H}$ , then there is a holomorphic function  $\Phi : \mathcal{T}_{g,n} \rightarrow \mathbb{H}$  such that

- (i)  $(\Phi \circ \mathcal{D}_q)(\lambda) = \lambda$  for all  $\lambda \in \mathbb{H}$ ,



(ii)  $(\Phi \circ T)(Y) = \Phi(Y) + t$  for every  $Y \in \mathcal{T}_{g,n}$  for some  $t > 0$ .

The next construction is a holomorphic map from a poly-plane  $\mathbb{H}^k$  to  $\mathcal{T}_{g,n}$  associated to Teichmüller disks arising from rational J-S differentials.

Fix a marked Riemann surface  $S \in \mathcal{T}_{g,n}$  and let  $\phi$  be a rational J-S differential on  $S$ . Let  $h_1, \dots, h_k$  denote the heights, with respect to the singular flat metric  $|\phi|$ , of the corresponding annuli  $\Pi_1, \dots, \Pi_j$ . We assume that each  $h_j > 0$ .

Let  $\mathbb{H}^k$  denote the  $k$ -fold Cartesian product of  $\mathbb{H}$  and define  $\mathcal{F} : \mathbb{H}^k \rightarrow \mathcal{BD}_1(S)$  by

$$\mathcal{F}(\lambda) = \left( \frac{i - \lambda_j}{i + \lambda_j} \right) \frac{|\phi|}{\phi}$$

on each  $\Pi_j$ , where  $\lambda = (\lambda_1, \dots, \lambda_k)$  and  $i = \sqrt{-1}$ .

This gives rise to the poly-plane map

$$(13) \quad \mathcal{E} : \mathbb{H}^k \rightarrow \mathcal{T}_{g,n}$$

defined by  $\mathcal{E}(\lambda) = [\mathcal{F}(\lambda)]$ .

Note that  $\mathcal{E}$  is holomorphic and its restriction to the diagonal is the Teichmüller disk  $\mathcal{D}_q$ . It turns out that the quasiconformal map with dilatation  $\mathcal{F}(\lambda)$  is an affine map in suitable holomorphic charts (cf. (11)) and one can use this to prove the following crucial property of  $\mathcal{E}$ :

$$\mathcal{E}(\lambda + (t, \dots, t)) = (T \circ \mathcal{E})(\lambda),$$

where  $t > 0$  and  $T \in \text{MCG}(S)$  is as above.

If we let  $f = \Phi \circ \mathcal{E}$ , then the holomorphic map  $f : \mathbb{H}^k \rightarrow \mathbb{H}$  satisfies certain properties, arising from the defining properties of  $\Phi$  and the equation above. Moreover it turns out that one can classify all holomorphic maps from  $\mathbb{H}^k$  to  $\mathbb{H}$  satisfying these properties. Indeed, for  $k = 2$  (this suffices for the main result), one concludes that there exist  $\alpha_1, \alpha_2 > 0$  such that

$$f(\lambda_1, \lambda_2) = \alpha_1 \lambda_1 + \alpha_2 \lambda_2$$

for all  $(\lambda_1, \lambda_2) \in \mathbb{H}^2$ .

To summarize, one has the following criterion for the equality of the Kobayashi and Carathéodory metrics:

**Theorem 7.1.** *Let  $X$  be biholomorphic to a Riemann sphere with five punctures. Let  $q$  be a rational J-S differential on  $X$  that decomposes the surface into exactly two annuli swept out by the closed horizontal trajectories of  $q$ . Then  $d_{\mathcal{T}}^K$  and  $d_{\mathcal{T}}^C$  agree on the Teichmüller disk  $\mathcal{D}_q$  if and only if the function  $\Phi : \mathcal{E}(\mathbb{H}^2) \rightarrow \mathbb{H}$  given by*

$$\Phi(\mathcal{E}(\lambda_1, \lambda_2)) = \alpha_1 \lambda_1 + \alpha_2 \lambda_2,$$

*for all  $(\lambda_1, \lambda_2) \in \mathbb{H}^2$  can be extended to a holomorphic function  $\Phi : \mathcal{T}_{0,5} \rightarrow \mathbb{H}$ .*

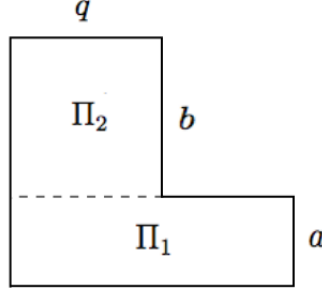


FIGURE 1. This L-shaped table determines a rational J-S differential with two cylinders.

### Step 2: Special Teichmüller disks in $\mathcal{T}_{0,5}$ .

The goal here is to construct certain Teichmüller disks in  $\mathcal{T}_{0,5}$  for which the criterion in Theorem 7.1 fails, to conclude that  $d_{\mathcal{T}_{0,5}}^K \neq d_{\mathcal{T}_{0,5}}^C$  on these disks. These special examples arise from *L-shaped pillowcases*: let  $a > 0, b \geq 0$  and  $0 < q < 1$  and let  $L(a, b, q)$  denote the L-shaped polygon in Figure 1. Let  $S = S(a, b, q)$  denote the double of  $L$ , as a Riemann surface. The form  $dz^2$  descends to a J-S differential  $\psi = \psi(a, b, q)$  on  $S$  and  $S$  decomposes into two non-degenerate annuli  $\Pi_1$  and  $\Pi_2$  swept out by the closed horizontal trajectories of  $\psi$ . The vertices of the polygon with interior angles  $\pi/2$  give rise to poles of order one on the doubled surface  $S$ , and these determine five marked points; the remaining vertex of angle  $3\pi/2$  determines a zero of order one for the resulting differential.

Fix  $(a_0, b_0, q_0) \in \mathbb{Q}^3$ . The J-S differential  $\psi_0 = \psi_0(a_0, b_0, q_0)$  is then rational. Let  $\mathcal{E}_0 : \mathbb{H}^2 \rightarrow \mathcal{T}_{0,5}$  denote the poly-plane map corresponding to  $\psi_0$  as in (13). It can be seen that

$$\mathcal{E}_0\left(\frac{b}{b_0}, \frac{a}{a_0}\right) = S(a, b, q_0)$$

for all  $a, b > 0$ .

The above equation combined with Theorem 7.1 and a continuity argument then shows the existence of a holomorphic function  $\Psi : \mathcal{T}_{0,5} \rightarrow \mathbb{H}$  such that

$$(14) \quad \Psi(a, b, q_0) = (a + bq_0)i$$

for all  $a > 0, b \geq 0$ .

Now fix  $q_0 \in (0, 1)$  and suppose that  $d_{\mathcal{T}_{0,5}}^K = d_{\mathcal{T}_{0,5}}^C$  on the Teichmüller disk  $\mathcal{D}_{\psi(a_0, b_0, q_0)}$  for some  $a_0, b_0 > 0$ .

The final and perhaps the most delicate part of the argument involves the smooth ( $C^\infty$ ) path

$$S(t) = S(a_0, 0, q_0 - t)$$

in  $\mathcal{T}_{0,5}$  with  $t > 0$  sufficiently small. The contradiction arises from the following

**Claim:** The composition  $\Psi \circ S$  is not smooth at  $t = 0$ .

The proof of this claim is based on the following expansion:

$$\Psi(S(a, 0, q-t)) = \Psi(S(a, 0, t)) + \beta_1(1+o(1))\frac{t}{\log t^{-1}} + \beta_2(1+o(1))\frac{t^2}{\log t^{-1}} + o\left(\frac{t^2}{\log t^{-1}}\right)$$

for some  $\beta_1$  and  $\beta_2 \neq 0$ . This is obtained by an application of (14) and some involved calculations of Schwarz-Christoffel maps for  $L$ -shaped polygons.

## 8. LOCAL CONVEXITY OF TEICHMÜLLER DOMAINS

While Markovic's result settles the question of convexity of Teichmüller domains, it turns out that one can extract finer information about such domains. In particular, one can show that no boundary point can be locally strictly convex, which we stated as Theorem 6.3. In this section we shall outline the proof of this result.

The strategy of the proof is inspired by K.-T. Kim's proof of the following result:

**Theorem 8.1** (Kim [22]). *For  $g \geq 2$  the Bers embedding of  $\mathcal{T}_g$  is not convex.*

As in Kim's proof, our proof involves two distinct components. Let  $\Omega \subset \mathbb{C}^{3g-3}$  be a Teichmüller domain with a locally strictly convex boundary point  $p \in \partial\Omega$ .

**Step 1.** This is the main part of [18]. The goal is to show that any point  $p$  as above is an orbit accumulation point for  $\text{Aut}(\Omega) = \text{MCG}(S)$ .

In Kim's work, one has the corresponding result for the Bers domain: *every* point of the Bers boundary is an orbit accumulation point. This follows from McMullen's result that cusps are dense in the Bers boundary ([34]), since powers of a Dehn twist converge to such cusps. Note that the orbit accumulation point property may not be preserved under biholomorphisms of domains as biholomorphism may not extend in an absolutely continuous manner (or indeed, even as a well-defined function!) to closures of domains.

**Step 2:** This step consists of *rescaling* at a “smooth” orbit accumulation point to obtain an unbounded convex domain  $\Omega_\infty$  in the limit with the following properties: (i)  $\Omega_\infty$  is biholomorphic to  $\Omega$  and (ii)  $\text{Aut}(\Omega_\infty)$  contains a one-parameter subgroup. These lead to a contradiction: Royden's theorem (Corollary 4.7) and (i) together imply that  $\text{Aut}(\Omega_\infty)$  is the discrete group  $\text{MCG}(S)$ .

**Localization.** The second step requires the technique of “localization” or complex rescaling, which is a simple and powerful method in complex analysis pioneered by S. Pinchuk ([41]), typically used to analyze the boundary mappings behaviour of holomorphic mappings.

We briefly discuss the notion of smoothness needed:

If  $\Omega$  is a domain in  $\mathbb{C}^N$ , we say that a point  $q \in \partial\Omega$  is *Alexandrov smooth* if  $\partial\Omega$  is the graph, near  $q$ , of a function which has a second-order Taylor expansion at

$q$ . It is a theorem of A.D. Alexandrov that almost every point of  $\partial\Omega$  is smooth in this sense if  $\Omega$  is convex. In particular, if  $\Omega$  is locally convex at  $q$  then we can assume that  $q$  is Alexandrov smooth, without loss of generality.

The original rescaling argument of Pinchuk requires the  $C^2$ -smoothness of the boundary of the domain under consideration. However, in Frankel's setting (for his proof of Theorem 6.1) or the case of the Bers domain, the boundary is far from being smooth. In his work Frankel introduced a rescaling technique along an orbit of the automorphism group accumulating at a boundary point which dispenses with the  $C^2$ -smoothness assumption. Subsequently, Kim and Krantz ([23]) developed a variant of Pinchuk's method in the convex case which does not require a  $C^2$ -smooth boundary and proved that, in fact, this recovers Frankel's rescaling. In brief, they show the following: Suppose  $\Omega$  is a domain, locally convex at an Alexandrov smooth point  $q \in \partial\Omega$ . Suppose that  $p_j = \gamma_j(p) \rightarrow q$  for  $\gamma_j \in \text{Aut}(\Omega)$  and  $p \in \Omega$ . Then there exist invertible complex affine transformations  $A_j : \mathbb{C}^N \rightarrow \mathbb{C}^N$  such that the biholomorphic embeddings  $A_j \circ \gamma_j : \Omega \rightarrow \mathbb{C}^N$  converge, on compact sets, to an embedding  $\psi : \Omega \rightarrow \mathbb{C}^N$ . Moreover, the convex domains  $A_j \circ \gamma_j(\Omega)$  converge in the local Hausdorff sense (The local convexity at a boundary point is crucial to obtain a subconvergent sequence of domains.) The limiting image  $\phi(\Omega)$  is convex and contains an affine copy of  $\mathbb{R}$  in its boundary. This immediately implies  $\mathbb{R} \subset \text{Aut}(\Omega)$  where the elements act by translations.

**Orbit accumulation point.** The proof of the orbit accumulation property in the first step is based on two basic results in Teichmüller theory:

First, the abundance of *Teichmüller disks*, that are complex geodesics (as defined in §4.2) in Teichmüller space. More precisely, by Theorem 4.2, one knows that through every point  $p \in \Omega$  and every direction  $v \in T_p\mathcal{T}_g$ , there exists a Teichmüller disk  $\tau : \Delta \rightarrow \mathcal{T}_g$  with  $\tau(0) = p$  and  $\tau'(0) = tv$  for some  $t > 0$ .

Second, Theorem 4.8 about the ergodicity of the Teichmüller geodesic flow due to Masur [32] and Veech [50]. Their work implies, in particular, that for any Teichmüller disk, almost every radial ray gives rise to a geodesic ray in  $\mathcal{T}_g$  that projects to a dense set in moduli space  $\mathcal{M}_g := \mathcal{T}_g/\text{MCG}(S)$ . In particular, there is a sequence of points along such rays that recur to any fixed compact set in  $\mathcal{M}_g$ .

Given these facts, the proof involves an elementary but delicate analysis of the boundary behaviour of holomorphic functions on the unit disk in  $\mathbb{C}$ . We choose a point  $p' \in \Omega$  which is close to the strictly convex point  $p \in \partial\Omega$  and take a complex geodesic  $\tau : \Delta \rightarrow \Omega$  with  $\tau(0) = p'$ . Since the boundary point  $p$  is locally strictly convex, there is a pluriharmonic “barrier” function  $h$ , namely the height from a supporting hyperplane at  $p$ , whose sub-level sets nest down to the single point  $p$ . This allows us to “trap” the complex geodesic  $\tau$ , by proving the existence of a positive measure set of radial directions in  $\Delta$  which under the holomorphic map  $\tau$  limit to boundary points arbitrarily close to  $p$ . One can then apply the

Masur-Veech ergodicity result to infer the existence of an orbit points shadowing such a radial ray, which accumulate arbitrarily close to  $p$ .

This concludes our discussion of the proof of Theorem 6.3.

### Further questions.

A natural question is whether one can relax the strict convexity assumption above to convexity:

**Conjecture 3.** *The Teichmüller space of a closed surface of genus  $g \geq 2$  cannot be biholomorphic to a bounded domain  $\Omega \subset \mathbb{C}^{3g-3}$  that is locally convex at some boundary point.*

It would be interesting to see if the analysis in Step 1 producing the orbit accumulation points can be extended to understand the statistics of the distribution of accumulation points on the boundary; even for the Bers domain, this would be a step towards a better understanding of the structure of the boundary.

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