Foliations by Minimal Surfaces (draft)

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1 Background

1.1 Introduction

In this article we shall examine the question of finding one-parameter families of minimal surfaces in a hyperbolic 3-manifold which is topologically a surface bundle over the line or the circle. Surfaces have proved useful in probing the geometry of such manifolds - in particular, families of pleated/$CAT(-1)$ surfaces, which share many properties of minimal surfaces, were tools in resolving the Ending Lamination Conjecture and the Tameness Conjecture.
The specific motivating question for us is

Question 1.1. *Is there an example of a hyperbolic 3-manifold fibering over the circle with the property that all of its fibers are minimal surfaces?*

It is result of Sullivan that there exists *some* metric on the fibered manifold such that all the fibers are minimal, so a related broader question is

Question 1.2. *What is the space of metrics on a 3-manifold fibering over the circle for which all the fibers are minimal surfaces? Are there such metrics with negative Ricci curvature?*

The subquestion is motivated by the fact that there are no known examples of negatively curved manifolds with a foliation by minimal surfaces, though there is no known differential-geometric obstruction. Note that asking for *constant* negative Ricci curvature metric reduces to the earlier question.
In the hyperbolic case, the only known obstruction to such a foliation is the presence of short geodesics, by an unpublished argument attributed to Thurston and Rubinstein. A 'positive' result in this setting is a recent result of Wang that shows the existence of constant mean curvature (CMC) foliations in certain quasi-fuchsian manifolds. This raises

Question 1.3. *Is there an example of a finite-volume fibered hyperbolic 3-manifold where the fibers are constant mean curvature surfaces?*

In this article we shall describe what is known, discuss some possible approaches, and list further questions that they have led to. The exploration of these questions tests our finer understanding of the hyperbolic structure in such manifolds, a lot of which has been understood from the point of view of coarse geometry in recent years.

1.2 Preliminaries

Here we shall list some of the definitions and facts together with notation used throughout this article.

Definition 1.4. The *second fundamental form* B of an immersion $i : \Sigma \to M$ of a surface in a Riemannian 3-manifold is a bilinear form on the tangent space at each point, defined as

$$B(X, Y) = \langle \nabla_X Y, \nu \rangle \tag{1}$$

where $X$and Y are vectors tangent to the surface, ν is the unit normal to the surface, and ∇ is the covariant derivative from the metric on M. The intrinsic *Gaussian curvature* of Σ is $k_1 k_2$ and the *mean curvature* is $k_1 + k_2$, where the *principal curvatures* k_1, k_2 are the eigenvalues of B.

Definition 1.5. A surface Σ immersed in $\mathbb{R}^3$, or more generally, in a 3-manifold M is a *minimal surface* is it has mean curvature H zero, or equivalently, is the critical point of the area functional.

Recall that the first and second variation formulae for a normal variation given by the function f are

$$\frac{\partial \mathcal{A}_t}{\partial t} = \int_\Sigma f \langle H, \nu \rangle dA \tag{2}$$

$$\frac{\partial^2 \mathcal{A}_t}{\partial t^2} = \int_\Sigma |\nabla f|^2 - f(\|B\|^2 + Ric(\nu,\nu))dA \tag{3}$$

where B is the second fundamental form, and $Ric(\nu)$ is the Ricci curvature of the ambient manifold M in the direction ν normal to the surface.

Definition 1.6. A *stable* minimal surface has second variation of area non-negative. A surface with second variation zero is *weakly* stable.

A general reference for minimal surfaces is the survey [CM99].

Definition 1.7. A *foliation* of a 3-manifold M by surfaces, is a family of disjoint, embedded surfaces or *leaves* whose union is the whole of M, such that locally, a neighbourhood of each point is diffeomorphic to an open set in $\mathbb{R}^3$ foliated by the family of planes $\mathbb{R}^2 \times \{t\}$. A foliation is *taut* if there is a closed loop transversal to each leaf.

We shall sometimes use the term 'minimally foliated' or $\mathcal{MF}$ to describe a foliated 3-manifold where each leaf is a minimal surface, and call that a *minimal foliation*. The manifolds of specific interest to us are hyperbolic.

Definition 1.8. A *hyperbolic* 3-manifold M has a complete Riemannian metric of constant sectional curvature -1, and is a quotient $\mathbb{H}^3/\Gamma$ where Γ is a *Kleinian group*, a discrete torsion-free subgroup of $PSL_2(\mathbb{C})$, the isometries of hyperbolic space.

A *quasi-fuchsian* hyperbolic manifold is topologically a product $S \times \mathbb{R}$ and has a compact convex 'core'. A hyperbolization theorem due to Thurston (see [Thu98]) is

Theorem 1.9. *(Thurston) A 3-manifold fibering over the circle admits a complete hyperbolic metric if and only if the the monodromy is a pseudo-Anosov surface diffeomorphism.*

Here we also note some geometric evolution equations (see [HP]) for the family of surfaces Σ_t in a hyperbolic manifold obtained by a *normal variation* $f\nu$ where $f : \Sigma_0 \to \mathbb{R}$ is a positive function and ν is the normal to the surface:

$$\frac{\partial g_{ij}}{\partial t} = 2fb_{ij} \tag{4}$$

$$\frac{\partial \nu}{\partial t} = -\nabla f \tag{5}$$

$$\frac{\partial b_{ij}}{\partial t} = -\nabla_i\nabla_j f + f(b_{ik}b^k_j + 1) \tag{6}$$

$$\frac{\partial H}{\partial t} = -\Delta f - f(\|B\|^2 - 2) \tag{7}$$

where $B = \{b_{ij}\}_{1\le i,j\le 3}$ is the second fundamental form of Σ_0.

The following definitions will be useful in later sections:

Definition 1.10. A *calibration* on a foliated Riemannian 3-manifold (M, g) is a closed 2-form ω such that (i) $\|\omega\|_g \leq 1$ and (ii) ω restricts to the area form on each leaf.

Theorem 1.11. *([HL82]) The existence of a calibration implies that each leaf is minimal, and in fact area-minimizing in its homology class.*

Proof. Choose a compact region D on a leaf and a homologous region D' sharing the same boundary. Then

$$Area(D) = \int_D \omega = \int_{D'} \omega \leq \int_{D'} \|\omega\| \leq \int_{D'} 1 = Area(D') \tag{8}$$

where the second equality is by Stokes theorem. □

Definition 1.12. The *Hopf differential* associated with a minimal (or constant mean curvature) immersion is the quadratic differential $A = \phi dz^2$ such that on a coordinate chart $\phi = \frac{1}{2}(b_{11} - b_{22} + 2ib_{12})$, where $\{b_{ij}\}_{1\leq i,j\leq 2}$ is the second fundamental form.

From the Gauss-Codazzi equations, it follows

Theorem 1.13. *For a constant mean curvature immersion in a space form (eg. $\mathbb{H}^3$), the associated Hopf differential is a holomorphic quadratic differential.*

1.3 Known results

Here we mention some relevant results which were important in motivating this study.
For compact manifold, we have the following

Theorem 1.14. *([SU82]) If S is an embedded $\pi_1 - injective$ surface in a compact manifold M, there exists an embedded minimal surface in its homotopy class.*

The existence of minimal surfaces in the non-compact quasi-fuchsian setting follows from the more general solution to the asymptotic Plateau problem by Anderson:

Theorem 1.15. *([And82]) Let Γ be a quasi-Fuchsian Kleinian group, and $K \subset \partial\mathbb{H}^3$ be a Γ-invariant Jordan curve at the boundary at infinity. Then there exists a Γ-invariant absolutely area-minimizing minimal plane whose asymptotic boundary is K.*

Question 1.1 was asked in an article by Uhlenbeck in which she proves, amongst other things, the following

Theorem 1.16. *([Uhl83]) For 'almost Fuchsian' quasi-Fuchsian manifolds, there is a minimal surface representative of the surface fiber whose principal curvatures k_i satisfy $|k_i| < 1, i = 1, 2$. Moreover, the minimal surface is unique in its homotopy class.*

The equation arising from the Gauss equation associated to a minimal immersion in a hyperbolic manifold is:

$$\Delta u = -\|A\|^2 e^{2u} + e^{2u} - 1 \tag{9}$$

where Δ is the Laplace operator on the surface with induced metric $g = e^{2u} g_h$ where g_h is the hyperbolic metric in its conformal class, and A is the Hopf differential.

Theorem 1.17. *([Uhl83]) For a 'sufficently small' Hopf differential A, satisfying $\|A\lambda^{-2}\|_\infty < 1$, there is an immersion of a complete minimal surface in $\mathbb{H}^3$ with a conformally flat metric of conformal factor λ, that induces A.*

Theorem 1.18. *([Cos04]) In a quasi-fuchsian manifold, there is a 'mean convex hull' MC contained in the convex core that contains all minimal surfaces.*

A natural question is

Question 1.19. *Given a quasi-fuchsian manifold, what is the 'size' of MC relative to the convex core? In particular, is there a sequence of manifolds for which the convex core gets larger and larger, while the mean convex hull remains bounded?*

The above theorem shows, for instance, that quasi-fuchsian manifold cannot have all its fibers minimal surfaces. However, recently Wang showed using the volume-preserving mean curvature flow that

Theorem 1.20. *([Wan08]) Uhlenbeck's almost-Fuchsian manifolds have a constant mean curvature foliation by surfaces in the homotopy class of the fiber.*

There are examples of quasi-fuchsian manifolds without any CMC foliation. For minimal foliations in doubly degenerate or 3-manifolds fibering over the circle, a non-existence result is as follows

Theorem 1.21. *(Thurston, Rubinstein)There is an $\epsilon > 0$ such that if a hyperbolic manifold contains a geodesic of length less than ϵ, then it cannot have a minimal foliation.*

Moving away from hyperbolic manifolds to arbitrary Riemannian manifolds, we have the following

Theorem 1.22. *([Sul79]) For a manifold with a taut foliation, there exists a Riemannian metric in which each leaf is a minimal surface.*

Note that the foliation by fibers in a 3-manifold M fibering over the circle is taut. In this setting, one can define the *mean curvature function* to be the function $f : M \to \mathbb{R}$ whose value is the mean curvature of the fiber passing through the point. Oshikiri used the above result to a more general result:

Theorem 1.23. *([Osh90]) Any smooth function on M that attains both strictly positive and strictly negative values is the mean curvature function for some Riemannian metric on M.*

We would be interested in curvature properties of these 'Sullivan metrics'. A relevant non-existence result is the following

Theorem 1.24. *(Schoen-Yau) A 3-manifold with positive scalar curvature cannot have an immersed stable minimal surface of genus $g > 1$.*

2 Main Results

In this section we shall describe some approaches to the previous questions and give some partial results and questions raised. Although some of the results could apply to a 3-manifold with a taut foliation, we shall be dealing mostly with compact 3-manifolds fibering over the circle, with pseudo-Anosov monodromy (see Thm 1.9).

2.1 Basic facts

Let (M, g) be a compact 3-manifold fibering over S^1, parametrized by t, with fiber Σ and projection to the circle π, with g a hyperbolic metric. Note that the last condition implies that the monodromy $\phi : \Sigma \to \Sigma$ is a pseudo-Anosov homeomorphism.
Assume the foliation $\mathcal{F}= \{\Sigma_t\}$ where $\Sigma_t = \pi^{-1}(t)$ is a minimal foliation.

Let $\{E_1, E_2, N\}$ be an orthogonal frame at each point of M, where E_1, E_2 span the tangent space of the leaf passing through that point.
Define the following 2-form

$$\omega(V_1, V_2) = det(\langle E_i, V_j\rangle_{1\leq i,j\leq 2}) \text{ for } V_i \in TM \tag{10}$$

Lemma 2.1. *ω is a calibration on M.*

Proof. This is a calculation - in general, the following is true:

$$d\omega = -2H dV_g = div_g(N) dV_g \tag{11}$$

where dV_g is the volume form on (M, g).
For a minimal foliation, $H \equiv 0$ so the form is closed. Also, by definition, $\omega|\mathcal{F}$ is the area element on the leaves. Geometrically, given two vectors, ω is measuring the area element spanned by their projection to the foliation. □

Lemma 2.2. *Each leaf in $\mathcal{F}$ is a (weakly) stable minimal surface.*

Proof. This follows from the previous lemma and Theorem 1.11. An alternate proof uses one of the geometric evolution equations for a normal variation mentioned earlier (7). For a minimal foliation,

$$0 = \frac{\partial H}{\partial t} = \Delta f - (\|B\|^2 - 2)f \tag{12}$$

where fN is the variation vector field normal to the surface, and B is its second fundamental form.
Details □

Corollary 2.3. *The area of each fiber Σ_t is the same.*

Proof. Consider a leaf-preserving variation. The first variation of area vanishes because of minimality, and the second variation vanishes because of weak stability. Hence the area is constant along the leaves. □

In fact, something slightly stronger than weak stability is true

Lemma 2.4. *Each leaf Σ_t is strictly area minimizing on proper compact subsets.*

Proof. By an argument in [FCS80], one can show that for a non-compact surface Σ, if we take the second-order partial differential operator of the form $L = \Delta + q$, we have that if there exists a positive function f satisfying $Lf = 0$, then the first eigenvalue of L is positive on each bounded domain in Σ. Since the second variation of area is a differential operator of the form L, the lemma follows. □

Lemma 2.5. *$\mathcal{F}$ is not invariant under the flow determined by N.*

Proof. Assume it is invariant. Then there are contradictions both locally and globally.
Locally, the geometric evolution equation (7) for H under the normal variation by $f \equiv 1$ reduces to

$$0 = \|B\|^2 - 2 \tag{13}$$

Hence the minimal surface has principal curvatures $k_1 \equiv 1, k_2 \equiv -1$, and is intrinsically a surface of constant negative curvature which is not totally geodesic, which is impossible. *Why?*
Globally, we have a Riemannian foliation with minimal leaves which is also impossible in a compact hyperbolic manifold (see [KW92]). □

Note that from the fact that in a negatively curved manifold, a geodesic is unique in its homotopy class, it follows

Lemma 2.6. *$\mathcal{F}$ is not invariant under any one-parameter family of isometries of M.*

Lemma 2.7. *For each surface Σ_t, the magnitude of its principal curvatures takes values strictly greater and strictly less than* 1*.*

Proof. If the principal curvatures of a leaf satisfy $|k_i| \leq 1$ then it cannot be part of a minimal foliation by Thm 1.16. If $|k_i| > 1$ everywhere then observe that the norm of the second fundamental form $\|B| > 2$ and hence by equation 12 we get a contradiction by the Maximum Principle. □

Lemma 2.8. *There is a universal constant $c > 0$ independent of the manifold M such that each minimal leaf Σ_t has principal curvatures bounded above by c.*

We close this section with a justification for concentrating on the fibered case:

Lemma 2.9. *A compact hyperbolic manifold that is also minimally foliated by closed surfaces is fibered over the circle.*

Proof. Since each closed minimal leaf is intrinsically negatively curved, there exists an area bound by the Gauss-Bonnet theorem. Then the result follows from Thm 5.5 of [And83]. Roughly, given a minimal leaf, there exists a product neighbourhood of definite size foliated by minimal surfaces. Patching up these products gives a surface bundle over a circle. □

2.2 Metric flow approach

The starting point of this approach is the observation that the fibered manifold M as in the previous section, has a *singular Sol* metric in which the fibers are *singular* minimal surfaces.

2.2.1 Smoothening singular Sol

Definition 2.10. A *singular flat* metric on a surface is a metric which is smooth and Euclidean except a discrete set of *cone points* which have a neighbourhood where the metric, in polar coordinates, is of the form

$$ds^2 = dr^2 + r^2 \left(\frac{\alpha}{2\pi}\right)^2 d\theta^2 \tag{14}$$

where α is the *cone angle*.

Recall that for a pseudo-Anosov surface homeomorphism $\phi : \Sigma \to \Sigma$ there is a singular flat metric on Σ such that in a local x, y-coordinate chart, ϕ is given by $\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$. The *Sol* metric on $\mathbb{R}^3$ is given by

$$ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2 \tag{15}$$

The *singular Sol* metric on $\mathbb{R}^3$ is a collection of k Sol half-spaces glued together along the t-axis, which forms the singular locus with cone angle $k\pi$.

Definition 2.11. A *singular Sol* metric on a 3-manifold M is a metric which is locally isometric to some singular Sol metric on $\mathbb{R}^3$.

Theorem 2.12. *(Cannon-Thurston [CT07]) If M is a Σ-bundle over S^1 with pseudo-Anosov monodromy ϕ, then M has a singular Sol metric such that each fiber Σ_t is given locally by $t =$ constant and hence acquires a singular flat structure. Moreover the monodromy is given locally by $(x, y, t) \mapsto (e^{-2t}x, e^{2t}y, t + 1)$.*

It is an easy calculation to verify that in the Sol metric, the horizontal planes $t =$ *constant* are minimal surfaces. Hence

Observation. In the singular Sol metric on M, each fiber Σ_t is minimal away from the singular locus.

Away from the singular locus this metric has negative sectional curvatures in two directions, but a positive sectional curvature in the plane tangent to the fibers.

Each fiber acquires an induced *flat* metric with isolated cone-singularities, with cone angle exceeding 2π. One can think of the negative curvature as being concentrated at these points for each fiber, and along the singular locus for the manifold M.
The idea then, is to somehow diffuse out this negative curvature from the singular locus, keeping the leaves minimal, till the manifold M acquires a smooth metric of (perhaps constant) negative curvature.

The main result of this section is to achieve a certain smoothening 'diffusion' of the singular Sol metric to a tubular neighbourhood of the singular locus, preserving the minimality of the fibers.

We shall begin with some calculations involving calibrations, in local coordinates.
Let ω be a calibration given locally by

$$\omega = \phi_1 dx + \phi_2 dy + \phi_3 dz \tag{16}$$

We shall assume, for convenience that $\phi_1 \equiv 1$.
Then the kernel of ω is the vector field

$$N = \phi_2 \partial_x - \phi_3 \partial_y + \partial_z \tag{17}$$

Then a metric $\{g_{ij}\}_{1\leq i,j,\leq 3}$ in which this N is orthogonal to the xy-planes has

$$g_{13} = -\phi_2 \ , \ g_{32} = \phi_3 \tag{18}$$

Using these, we have the following statement for open sets in $\mathbb{R}^3$

Lemma 2.13. *If the metric is of the form*

$$\begin{pmatrix} 1 & 0 & f \\ 0 & 1 & g \\ f & g & h \end{pmatrix}$$

then there is a calibration ω and the intrinsically Euclidean xy planes are minimal if the following equation holds

$$f_x = -g_y \tag{19}$$

Proof. The calibration is given by

$$\omega = dxdy - f dydz + g dxdz \tag{20}$$

Equation (19) implies that ω is closed. □

Similarly, in cylindrical coordinates,

Lemma 2.14. *If the metric is of the form*

$$\begin{pmatrix} 1 & 0 & f \\ 0 & r^2 & g \\ f & g & h \end{pmatrix}$$

then there is a calibration ω and the intrinsically Euclidean $r\theta$ planes are minimal if the following equation holds

$$\frac{1}{r}\frac{\partial g}{\partial \theta} = -f - r\frac{\partial f}{\partial r} \tag{21}$$

Moreover, the pair

$$f(r,\theta) = -\frac{f_1(r)}{r}\cos(2\theta)\ ,\ g(r,\theta) = \frac{r f_1'(r)}{2}\sin(2\theta) \tag{22}$$

are solutions to (21) for any smooth single-variable function f_1.

Note that via the 'height-preserving' homeomorphism $(x, y, z) \mapsto (e^{-2z}x, e^{2z}y, z)$ the Sol metric on $\mathbb{R}^3$ pulls back to a metric given by the matrix

$$\begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & y \\ -x & y & 1+x^2+y^2 \end{pmatrix}$$

It is interesting to note that this is invariant of z - a consequence is in Sol, there is a Killing field (in these coordinates, the vertical vector field) which is *not* orthogonal to any foliation.
In cylindrical coordinates, it can be checked that this metric is

$$\begin{pmatrix} 1 & 0 & -r\cos(2\theta) \\ 0 & r^2 & r^2\sin(2\theta) \\ -r\cos(2\theta) & r^2\sin(2\theta) & 1+r^2 \end{pmatrix}$$

By the previous two lemmas, there is an associated calibration that makes the $r\theta$-planes minimal, which is just the pullback of the calibration $dx \wedge dy$ in usual coordinates of Sol.

We shall denote Sol in these coordinates as Sol_{inv}. The advantage of these coordinates is that now in the singular Sol_{inv} metric, the $r = a$ locus in the $z = 0$ half-planes is taken to the $r = a$ locus in the $z = 1$ half-planes by the pseudo-Anosov monodromy (see Thm 2.12). The $r = a$ solid cylinder glues up to give a tubular neighbourhood C about the singular locus, and it is here that we perform our 'smoothening'.

The chief ingredient that we use is the following extension of a special case of the Gromov-Thurston '2π-theorem' (see [GT87]):

Lemma 2.15. *Let M be an 3-manifold with torus boundary and a Riemannian metric which in a collar neighbourhood of the torus is negatively curved in the 'longitudinal' direction and non-positively curved in the remaining directions. If, moreover, the meridian has length greater than 2π, then the torus can be 'filled' by solid torus and the resulting manifold given a complete negatively curved metric.*

To be able to apply this, we need:

Lemma 2.16. *We can construct fibered* 3*-manifolds* M *with the singular Sol metric having singular locus with one component and cone angle greater than* 2π*, and an embedded neighbourhood of radius greater than* 1*.*

Proof. We can have singular flat metrics on a surface with a single cone point, and we can ensure a definite neighbourhood by passing to a finite cover. □

Hence we can assume our solid torus neighbourhood C of the singular locus has radius $r = a > 1$.
We divide C into the smaller radius-1 neighbourhood C_1 and the 'annular' region $C_2 = C \setminus C_1$.
In the region C_2, we smoothly interpolate between the Sol_{inv} metric at $r = a$ and the following negatively curved metric at $r = 1$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 + r^2 \end{pmatrix}$$

For the interpolation, we use a metric of the form

$$\begin{pmatrix} 1 & 0 & -\frac{f_1(r)}{r}\cos(2\theta) \\ 0 & r^2 & \frac{r f_1'(r)}{2}\sin(2\theta) \\ -\frac{f_1(r)}{r}\cos(2\theta) & \frac{r f_1'(r)}{2}\sin(2\theta) & 1 + r^2 \end{pmatrix}$$

since by lemma 2.14 the $r\theta$ planes will remain minimal in this metric.
This is easily done since one can construct a function f_1 that smoothly interpolates between the constant 0 function on $r \leq 0$ and the function $r \mapsto r^2$ on $r \geq a$.

Notice that the region C_1 has a meridian of length equal to the cone angle about the singular locus (adding up the lengths of the half-circles in the glued half-planes) which is greater than 2π. Then our extension of the 2π-theorem allows us to construct a new *smooth* negatively curved metric in C_1 of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & f^2(r) & 0 \\ 0 & 0 & g^2(r) \end{pmatrix}$$

Note that the $r\theta$-planes are totally geodesic in this metric, and intrinsically negatively curved.
This completes the construction of a smooth metric that agrees with the Sol metric outside C and keeps the leaves minimal.

This smoothening completes a first step towards a more general global flow which can perhaps lead to minimally foliated manifolds with desired curvature properties.

Question 2.17. *Is there a global geometric evolution of metrics on the manifold M that converges to a smooth negatively curved metric with all the fibers Σ_t being minimal surfaces?*

It would also be interesting to know

Question 2.18. *What are the analytical or geometric obstructions to the existence of such a flow?*

For instance,

Question 2.19. *Is the existence of short geodesics an obstruction to a minimal foliation for variable negatively curved metrics?*

In a subsequent section, we shall study some other deformations of metrics preserving a minimal foliation.

2.2.2 Leafwise Ricci flow

We shall describe an evolution of metrics on a Riemannian 3-manifold (M, g_0) with a transversely-oriented foliation $\mathcal{F}$ by compact surfaces that restricts to the Ricci flow on each leaf.
Let the foliation $\mathcal{F} = \{(L_\alpha, g_\alpha)\}$ where L_α is a leaf with the induced metric g_α, and let N denote the unit vector field normal to the foliation.

Lemma 2.20. *The family of metrics $\{g_\alpha\}$ and the vector field N together determine the metric g on M. The converse is also true, except for the ambiguity in the sign for N.*

Definition 2.21. The *leafwise Ricci flow* on $(M, \mathcal{F}, g_0)$ is a one-parameter family of metrics g_t on M determined by the pair $\{g_\alpha(t), N\}$ where $g_\alpha(t)$ satisfy the following evolution equations:

$$\frac{\partial g_\alpha}{\partial t} = (r - R) g_\alpha \tag{23}$$

where R is a function on the surface equal to twice the Gaussian curvature, and r is the the average of R over the surface.

The following is immediate from the definitions.

Lemma 2.22. *Let $\phi : (M, \mathcal{F}, g) \to (M', \mathcal{F}', g')$ be an isometry carrying leaves to leaves, where g and g' are determined by $(\{g_\alpha\}, N)$ and $(\{g'_\alpha\}, N')$ respectively. Then the manifolds $(M, \mathcal{F}, g_t)$ and $(M', \mathcal{F}', g'_t)$ determined by the leafwise Ricci flow for time $t > 0$ are also isometric via ϕ.*

Recall the main facts of Ricci flow for surfaces (see for instance [CCG+07]), which we shall summarize as the following theorem.

Theorem 2.23. *The Ricci flow on a surface L_α given by the above equation preserves the conformal class of the metric, and hence the solution is of the form*

$$g_\alpha(t) = e^{u(t)} g_\alpha(0) \tag{24}$$

where the function $u(t) : L_\alpha \to \mathbb{R}$ satisfies

$$\frac{\partial u}{\partial t} = r - R \tag{25}$$

Moreover, if the initial average curvature $r < 0$, then the flow exists for all time, and $g_\alpha(t)$ converges to a metric of constant curvature r.

We note in the construction of the foliated manifold M in the previous section, that the last condition of negative average curvature can be guaranteed for each leaf Σ_t. Recalling the notation of the previous section

Lemma 2.24. *For genus g large enough, one can choose an initial singular flat metric on the leaves Σ_t such that in the new metric obtained by 'smoothening' in the singular locus, the average curvature of each leaf is a constant $r < -1$.*

Proof. We construct the surface leaf by gluing sides of a $4g$-gon, with side length l and inradius d, where

$$d = \frac{l}{2\tan(\pi/4g)} \tag{26}$$

Note that to be able to carry out the construction in the previous section, it suffices to have $d > \frac{1}{2g-2}$ since by Gauss-Bonnet the total cone-angle is $2\pi(g-1)$ and we require some equidistant locus of length greater than 2π to be able to use the 2π-theorem.
Choosing $d \approx \frac{1}{2g-2}$, we have that the area of this flat surface

$$A_{flat} = 4g\tan(\pi/4g)d^2 \approx \frac{g\tan(\pi/4g)}{(g-1)^2} \tag{27}$$

which goes to a 0 as $g \to \infty$. It can be checked that the new metric A_{new} differs from A_{flat} by a constant independent of g.
By Gauss-Bonnet, the absolute value of the average (negative) curvature is

$$|r| = \frac{4\pi(g-1)}{A_{new}} \approx \frac{4\pi(g-1)}{O(1)} \tag{28}$$

which is large, and in particular greater than 1, for large g. ☐

Corollary 2.25. *One can construct smoothened singular Sol manifolds such that the leafwise Ricci flow exists for all time and converges to a metric where each leaf Σ_t has constant negative curvature, and the sectional curvature tangent to the fiber is negative.*

In general, the leafwise Ricci flow will not preserve the minimality of leaves. It follows immediately from Lemma 2.32 that

Lemma 2.26. *For a leafwise Ricci flow on a minimally foliated manifold M, let the conformal exponent for the metric at time t be $u(t) : M \to \mathbb{R}$. Then the foliation remains minimal if $N(u(t)) \equiv 0$, for $t > 0$ where N is the unit normal vector field.*

Proof. Let ω be the calibration for (M, g_0). Then $\omega(t) = e^{u(t)}\omega$ is a calibration for (M, g_t):
1. Restriction to leaves is the area form, since the area element μ satisfies

$$\frac{\partial \mu}{\partial t} = (r - R)\mu \tag{29}$$

which is the same as equation (25).
2. The kernel of $\omega(t)$ is still N.
3. $d\omega(t) = \omega(t) \wedge d(e^u(t)) + d(e^(u(t)) \wedge d\omega = \omega(t) \wedge d(e^u(t)) = 0$ since $d(e^(u(t))(N) =$ by the given condition. □

However in the case that we are interested in,

Lemma 2.27. *For the smoothened singular Sol manifolds,*

$$N(u(0)) \equiv 0, \text{ and } \frac{\partial}{\partial t}_{|0} N(u(t)) \equiv 0 \tag{30}$$

but $N(u(t))$ is not identically zero for all time.

The only known methods of showing the existence of a negatively curved metric on surface bundles over the circle with pseudo-Anosov monodromy are Thurston's hyperbolization theorem and Perelman's Ricci flow with surgeries.

Question 2.28. *Is there a way to evolve N simultaneously with leafwise Ricci flow to get an eventual metric with negative sectional curvatures?*

With regard to our main interest,

Question 2.29. *Is there a way to evolve N simultaneously with leafwise Ricci flow to get an eventual metric with a minimal foliation? For instance, is there a way to combine leafwise Ricci flow and the mean-curvature flow?*

More generally,

Question 2.30. *Is there a possible version of Ricci flow on the 3-manifold M with the added constraint of preserving minimality of leaves? For example, what are the possible energy functionals that could lead to such a geometric flow?*

2.2.3 Space of Sullivan metrics

We know from Sullivan's theorem that there exists a metric on M for which it is minimally foliated. In this section we shall recall two proofs of this fact, and investigate the space of such metrics which shall be denoted as $\mathcal{S}$. In particular, we shall exhibit certain infinite-dimensional families of paths within $\mathcal{S}$.

Theorem.(Sullivan) For a manifold with a taut foliation, there exists a Riemannian metric in which each leaf is a minimal surface.

Proof. Given the foliated manifold, it is enough to build a nowhere vanishing closed 2-form ω with $\omega|_{\mathcal{F}} \neq 0$. This gives a non-vanishing vector field N transverse to the foliation such that $\omega(N, .) = 0$,and one can construct a Riemannian metric for which N is a unit vector field orthogonal to the foliation and $\omega|_{\mathcal{F}}$ is the area form on the leaves. Then ω is a calibration, and the theorem follows.

Sullivan's approach was to show the existence of such a closed 2-form ω by using the Hahn-Banach theorem. Briefly, one considers the vector space of 1-currents on M and the cone of *foliation currents* generated by the Dirac currents tangent to the foliation. It can be shown that the tautness of the foliation is equivalent to this cone intersecting the subspace of 1-cycles only at the origin. The Hahn-Banach theorem gives a non-zero linear functional which vanishes on the cycles and is non-zero on the cone. By a duality due to Schwarz, this translates to the existence of a closed form ω as above.

An alternate more constructive approach due to Thurston and Hass (see [Has86]) is to first cover the manifold with solid tori with the core curves transverse to the foliation, and consider the form closed $dx \wedge dy$ in local coordinates, and patch them up using a partition of unity. The resulting form ω satisfies our requirements. □

Definition 2.31. A *Sullivan metric* is a Riemannian metric g on a manifold with a taut foliation such that each leaf is minimal.

Let g be a metric on the foliated manifold $(M, \mathcal{F})$ such that $\mathcal{F} = \{L_\alpha\}$ is a minimal foliation. Recall that it uniquely determines the unit normal vector field N and the induced metric on the leaves $\{g_\alpha\}$.
Let the vector field $X = \alpha N + F$ where $\alpha : M \to \mathbb{R}^+$ be a smooth positive function and F is a vector field tangent to the foliation. Let $g'_\alpha = e^u g_\alpha$ where $u : M \to \mathbb{R}$ is a smooth positive function.
Consider the metric g' determined by this new pair $(\{g'_\alpha\}, X)$ (as in the previous section).

Lemma 2.32. *This new metric g' is also a Sullivan metric if*

$$X(\log \alpha) - div_g X = N(u) \tag{31}$$

Proof. This is a lengthy calculation. We express the mean curvature in terms of the covariant derivative ∇ and use the following well-known formula for expressing them in terms of the metric g:

$$2g(\nabla_X Y, Z) = \partial_X(g(Y, Z)) + \partial_Y(g(X, Z)) - \partial_Z(g(X, Y))$$
$$+g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)$$

We then track the changes when g is replaced by g'. *[Add details]* □

Note that this gives some characterization of a *slice* of $\mathcal{S}$, since we are not changing the conformal class of the induced metric on each leaf. For a 3-manifold fibering over the circle with fiber Σ, this means that the loop in $Teich(\Sigma)$ recording the conformal structures of each leaf, is kept fixed.

Question 2.33. *What is a useful characterization of $\mathcal{S}$ in terms of deformations of the loop in $Teich(\Sigma)$ and the normal vector field N?*

Examining Sullivan's proof more carefully, we can show that for a 3-manifold M fibering over the circle,

Lemma 2.34. *There is a Sullivan metric h on M for which the unit normal vector field N gives an transverse foliation by geodesics.*

Proof. Let α be a closed 1-form that vanishes on $T\mathcal{F}$, tangent to the foliation. For example, $\alpha = \pi^*(dt)$, the pullback of dt by the submersion $\pi : M \to S^1$. By the argument in Sullivan's proof, there exists a closed 2-form ω such that $\omega|_{\mathcal{F}} > 0$.
We claim there exists a metric h such that $\alpha = *\omega$.
Let N' be the kernel of ω. Note that $\alpha(N) > 0$ since $N \in T\mathcal{F}^{\perp}$. Normalize to get $N = \frac{1}{\alpha(N')}N'$, and define the metric h such that N is a unit normal vector field, and the area form on $\mathcal{F}$ coincides with $\omega|_{\mathcal{F}}$. Hence locally, if $\omega = dx \wedge dy$, then $\alpha = dz$, which is the Hodge dual as desired.
This α is then a harmonic 1-form, and forms a calibration of the 1-dimensional foliation of the integral curves of N, which are therefore geodesics. □

Such a metric will be referred to as a *Katz metric* (see [Kat02]) and is a candidate for a 'canonical' Sullivan metric.

Question 2.35. *Is there a geometric flow of metrics in $\mathcal{S}$ that converges to a Katz metric?*

There are many unanswered questions about the space $\mathcal{S}$:

Question 2.36. *Is $\mathcal{S}$ a manifold?*

Question 2.37. *What are the metric properties of $\mathcal{S}$ with, say, the L^2-metric? Is it complete?*

2.3 Other approaches

We shall describe some other approaches that are more specific for hyperbolic 3-manifolds.

2.3.1 Taking finite covers

The starting point is Theorem 1.21 the proof of which involves using a deep Margulis tube to construct mean-convex surfaces that act as 'barriers' to minimal surfaces by the Maximum Principle, ruling out minimal foliations.
An alternate approach using shrinkwrapping leads to a recent result of Wang:

Theorem 2.38. *([Wan09]) Let M be a quasi-fuchsian hyperbolic 3-manifold containing a geodesic γ whose complex length is $l + i\theta$ where $\frac{\theta}{l} \gg 1$. Then there exists at least two minimal surfaces in M.*

A careful study of the proof leads to the following two propositions.

Lemma 2.39. *Wang's result can be extended to degenerate hyperbolic 3-manifolds also.*

Lemma 2.40. *If M satisfies the conditions as in the theorem, then it does not have a minimal foliation.*

Note that this is slightly weaker than Thm 1.21.

Question 2.41. *Can Wang's proof be extended to strengthen the condition to γ having hyperbolic length $l \ll 1$?*

The remaining case, therefore, is when there are no really short geodesics, that is, there is a lower bound to the injectivity radius of our fibered 3-manifold manifold M. The idea is to pass to finite covers where there *are* deep Margulis tubes about a closed geodesic, which is possible by the residual finiteness of $\pi_1(M)$ (see [Gab94]).
The issue, however, is that the genus of the fiber is also larger in that finite cover, and hence the required depth of the tube also increases.

Question 2.42. *Is there any other way to resolve this issue? More specifically,..*

Passing to finite covers is involved in the proof of the following theorem:

Theorem 2.43. *(Rubinstein)([Rub05]) In a finite-volume hyperbolic 3-manifold, there exists infinitely many distinct immersed minimal surfaces.*

Question 2.44. *Is there any other way of constructing 'barriers' that does not involve using a deep Margulis tube? For example, do high-distance Heegaard splittings lead to any similar constructions?*

2.3.2 Working in the universal cover

Passing to the universal cover, the incompressible fibers of the surface bundle M lift to embedded planes which limit onto the entire sphere at infinity. The idea is to study this starting from our understanding of the quasi-Fuchsian case, when the limit set is a Jordan curve and which leads to the degenerate case in the geometric limit.

Anderson's solution of the asymptotic Plateau problem (see Thm 1.15) shows the existence of at least one area-minimizing plane limiting onto a given quasi-circle. Coskunuzer studied the case when the limit set is a C^3 Jordan curve, and proved that generically but not always, such curves bound a unique area-minimizing plane (see [Cos06]).
Note that for a compact 3-manifold fibering over the circle, a minimal surface

representative of the fiber (that exists by Thm 1.14) lifts to infinitely many minimal planes in the universal cover with limit set the entire sphere, differing by a deck transformation by a hyperbolic isometry.

Question 2.45. *Is there a Jordan curve on $\mathbb{S}^2 = \partial\mathbb{H}^3$ which is the limit set of infinitely many area-minimizing planes?*

Given any minimal plane in $\mathbb{H}^3$, a foliation can be obtained by applying a 1-parameter group of hyperbolic isometries.

Question 2.46. *Are there examples of foliations of $\mathbb{H}^3$ by minimal planes which are not invariant by a 1-parameter group of isometries?*

Given an immersion of a minimal plane in $\mathbb{H}^3$, Velling ([Vel99]) studied the normal flow N_t, $t \in [0, \infty)]$ that in the limit gives the *hyperbolic Gauss map* to the sphere at infinity.
If g_t is the induced metric on the surface Σ_t, then the scaled metrics $e^{-2t} g_t$ converge to a metric g_∞ in the limit.

Theorem 2.47. *(Velling) If Σ_0 is a CMC immersion, then the map $N_t : \Sigma_0 \to \Sigma_t$ is harmonic, and so is the hyperbolic Gauss map $N_\infty : \Sigma_0 \to (\mathbb{S}^2, g_\infty)$.*

This is analogous to the situation in Euclidean space:

Theorem 2.48. *For a CMC immersion in $\mathbb{R}^n$, the Gauss map to the round sphere is harmonic. For a minimal immersion in $\mathbb{R}^3$, it is in fact anti-holomorphic.*

Question 2.49. *Can one assign an appropriate 'object at infinity' for a minimal plane in $\mathbb{H}^3$ such that the study of families of such objects could rule out the existence of a minimal foliation?*

A candidate for such an object is perhaps a complex projective structure induced via the hyperbolic Gauss map. It is known (see [Eps86]) for a plane limiting onto a Jordan curve, the hyperbolic Gauss map is a diffeomorphism onto the two complementary regions on the sphere at infinity if its principal curvatures satisfy $|k_i| < 1$.

Question 2.50. *Is there a useful version of the hyperbolic Gauss map for a more general plane? for a minimal plane?*

Velling had also studied the path in Beltrami differential space traced out by the normal flow.

Question 2.51. *For a minimal foliation, or more generally a CMC foliation, what are the possible paths in Teichmuller space, corresponding to the conformal structures of the leaves)?*

A calculation involving the evolution equations for normal variation (see Section 1.2) gives the following local statement (valid in some coordinate chart):

Lemma 2.52. *If μ_t is the Beltrami coefficient of the map $V_t : \Sigma_0 \to \Sigma_t$, where V is the normal variation by the function f on Σ_0. Let Σ_0 be conformally immersed with conformal factor λ and second fundamental form A, and let each Σ_t be a minimal surface. Then*

$$\frac{\partial \mu_t}{\partial t} = 4fA/\lambda \tag{32}$$

We also note the following observation obtained from techniques of a paper by Huang ([Hua07]):

Lemma 2.53. *In a 3-manifold topologically of the form $\Sigma \times \mathbb{R}$ with some product metric with the fibers having constant negative curvature and projecting to a Weil-Petersson geodesic in $Teich(\Sigma)$, the leaves are in fact minimal.*

In [Hua07], it is shown that such a manifold is *not* negatively curved.
Similar to the last question, we can also ask

Question 2.54. *For a minimal foliation, or more generally a CMC foliation, what are the possible paths in space of quadratic differentials (corresponding to the induced Hopf differentials on the leaves)?*

Velling's work on characterizing the Hopf differential A associated with minimal immersions of a plane in hyperbolic space, using equation (9) and some analysis leads to

Theorem 2.55. *If $\|A| > \frac{1}{2}$ everywhere, then A cannot be the Hopf differential of a minimal immersion.*

Question 2.56. *Is there a geometric proof of this?*

2.3.3 Other related issues

Harmonic 1-forms.

Definition 2.57. A *harmonic 1-form* β is a closed 1-form that satisfies the Laplace equation $\Delta\beta = 0$.

Suppose in a 3-manifold M fibering over the circle, with a Riemannian metric g, we have a nowhere-zero 2-form ω whose kernel is N. Let α be the Hodge dual of ω. Then we have the following relations:

Lemma 2.58. *α is harmonic $\implies$ ω is a calibration $\implies$ N is volume preserving.*
Furthermore, none of the reverse implications are true.

Hence, a non-singular harmonic 1-form would imply the existence of a minimal foliation. Note that Bochner's identity rules out the existence of the former in manifolds of positive Ricci curvature.

Minimal surfaces in Sol.
Part of the investigations involving calibrations led to constructing the following foliation $\mathcal{F}$ by minimal planes L_α in $\mathbb{R}^3$ with the Sol metric:

$$L_\alpha = \{x^2 + y^2 - 2z^2 = \alpha\} \tag{33}$$

Details

CMC foliations.
CMC foliations of homogeneous spaces have been studied in higher dimensions (see for example the survey [MPR]).
One of the conjectures raised in that survey is

Question 2.59. *(Meeks) Is the mean curvature function of any CMC foliation of $\mathbb{H}^2 \times \mathbb{R}$ bounded by $\frac{1}{2}$?*

One approach to Question 1.3 regarding CMC foliations of compact hyperbolic 3-manifolds is to first prove the existence of *some* geometrically well-controlled foliation, for example one with bounded principal curvatures or some sort of monotonicity of mean curvature that will allow a volume-preserving mean curvature flow to work.

Question 2.60. *Can CMC foliations of $\mathbb{H}^2 \times \mathbb{R}$ be studied using the volume preserving mean curvature flow?*

A dichotomy
In the context of minimal foliations in hyperbolic $S \times \mathbb{R}$, the following interesting dichotomy was suggested by Minsky:

Conjecture. If there exists such a manifold M which does not have any minimal surfaces representative of the fiber, then there exists another such manifold N which has a minimal foliation.

The idea of a proof is to use the mean curvature flow to 'flow out' the initial foliation towards a minimal one, and to 'rescale' the resulting foliated manifolds to converge to a new one with a minimal foliation in a geometric limit.

3 Future work

One of the difficulties with the question of existence of minimal foliations (Question 1.1) is that it does not seem to be directly approachable by any known machinery. The metric flow approach outlined in this article is an attempt to create tools to help with this question. However, given the analytical difficulties of similar but easier questions in geometric analysis, this seems to be hard to pursue. It must be mentioned, though, that the conjectural answer to Question 1.1 is 'no'.
This article raises several other approaches and questions which I hope to pursue. In particular, establishing a possible connection of minimal planes in $\mathbb{H}^3$

with complex projective structures via a suitably defined version of the hyperbolic Gauss map (Question 2.50) would be useful since it would allow a whole wealth of techniques to be used.

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