ASYMPTOTICITY OF GRAFTING AND TEICHMÜLLER RAYS II

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Abstract. We show that any grafting ray in Teichmüller space is (strongly) asymptotic to some Teichmüller geodesic ray. Given a grafting ray, we define its limiting surface, and a conformally equivalent singular-flat surface of infinite area that represents the limit of the desired Teichmüller ray. The proof involves building quasiconformal maps of low dilatation between the surfaces along the rays. Our preceding work had proved the result for rays determined by an arational lamination or a multicurve, and the unified approach here gives an alternative proof of the former case.

1. Introduction

Grafting rays and Teichmüller rays are one-parameter families of (marked) conformal structures on a surface of genus g, that is, they are paths in Teichmüller space $\mathcal{T}_g$, that arise in two disparate ways. On one hand, Teichmüller rays are geodesics in the Teichmüller metric on $\mathcal{T}_g$, that relate to the complex-analytic structures on Riemann surfaces. On the other, grafting is an operation that has to do with the uniformizing hyperbolic structures (we shall assume throughout that $g \geq 2$), or more generally, complex projective structures on a surface (see [KT92], [Tan97]). Both rays are determined by the data of a Riemann surface X which is the initial point and a measured lamination λ that determines a direction. In our preceding paper ([Gupa]) we proved a strong comparison between these two for a "generic" lamination in the space of measured laminations $\mathcal{ML}$, and here we extend that to the most general result.

Recall that two rays Θ and Ψ in $\mathcal{T}_g$ are said to be asymptotic if the *Teichmüller distance* (defined in §2) between them goes to zero, after reparametrizing if necessary. We shall prove:

Theorem 1.1. *Let $(X, \lambda) \in \mathcal{T}_g \times \mathcal{ML}$. Then there exists a $Y \in \mathcal{T}_g$ such that the grafting ray determined by (X, λ) is asymptotic to the Teichmüller ray determined by (Y, λ).*

In the preceding paper we had proved the above statement assuming λ was *arational* or was a multi-curve. This assumption was mild, since this includes a full-measure subset of $\mathcal{ML}$, and was enough for proving certain density results in moduli space (Corollary 1.2 and Theorem 1.5 of [Gupa]). In this paper, we remove any such assumption. This is the first step of a fuller comparison, in forthcoming work, between the dynamics of grafting and the much-studied Teichmüller geodesic flow ([Mas82], [Vee86] - see [Mas] for a survey). For some earlier work in this direction see [DK12] and the "fellow-travelling" result in [CDR12].

The strategy of the proof of Theorem 1.1 generalizes that for the multicurve case in [Gupa], and yields an alternative proof of the arational case. A key step is to consider a "generalized half-plane surface", an infinite-area singular flat surface comprising half-planes and half-infinite cylinders singular flat surfaces, that represents a limit of the singular-flat structures along a Teichmüller ray. Equivalently, this is a Riemann surface equipped with certain meromorphic quadratic differential with higher order poles (see the Appendix, and [Gupb]).

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In the multicurve case, this limiting surface is built from half-infinite Euclidean cylinders, and corresponds to a quadratic differential with poles of order two.

As discussed in the Appendix, the following result is an easy generalization of the main result of [Gupb]:

Theorem A.7. *For any Riemann surface Σ with a set of marked points P, there exists a meromorphic quadratic differential with prescribed local data (orders and residues) at P that induces a generalized half-plane structure.*

The proof of Theorem 1.1 proceeds by equipping the limit of the grafting ray with an appropriate quadratic differential using the above result, and reconstructing a Teichmüller ray that limits to that half-plane surface. The asymptoticity is established by constructing quasiconformal maps of low dilatation from (large) grafted surfaces to this ray. This requires building an appropriate decomposition of the surfaces, and adjusting the conformal map between the limits on each piece. This uses some of our previous work in [Gupa] together with some technical analytical lemmas.

Outline of paper. In §3 we recall some of the results from [Gupa] which we employ in this paper, and discuss the notion of "limits" of grafting rays. In §4 we provide a compilation of the technical lemmas involving quasiconformal maps. Following an outline of the proof in §5, we complete the proof of Theorem 1.1 in §6. The Appendix recalls some of the work in [Gupb] and outlines the generalization to Theorem A.7 that is used in §6.

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2. Preliminaries

2.1. **Teichmüller space.** The Teichmüller space $\mathcal{T}_g$ is the space of marked Riemann surfaces of genus g. More precisely, if S_g denotes a (fixed) topological surface of genus g, it is the collection of pairs:

$$\begin{aligned}\mathcal{T}_g = \{(f,\Sigma)|\ f: S_g \to \Sigma \text{ is a homeomorphism,}\\ \Sigma \text{ is a Riemann surface } \}/\sim\end{aligned}$$

where the equivalence relation is:

$$(f,\Sigma) \sim (g,\Sigma')$$

if there is a conformal homeomorphism $h:\Sigma \to \Sigma'$ such that $h \circ f$ is isotopic to g.

Teichmüller metric. The *Teichmüller distance* between two points X and Y in $\mathcal{T}_g$ is:

$$d_{\mathcal{T}}(X,Y) = \frac{1}{2}\inf_f \ln K_f$$

where $f: X \to Y$ is a quasiconformal homeomorphism preserving the marking and K_f is its quasiconformal dilatation. See [Ahl06] for definitions.

2.2. **Quadratic differential metric.** A *quadratic differential* q on $X \in \mathcal{T}_g$ is a differential of type $(2,0)$ locally of the form $q(z)dz^2$. It is said to be *holomorphic* (or *meromorphic*) when $q(z)$ is holomorphic (or meromorphic).

A meromorphic quadratic differential $q \in Q(X)$ defines a conformal metric (also called the q-metric) given in local coordinates by $|q(z)||dz|^2$ which is flat away from the zeroes and poles. At the zeroes, the metric has a cone singularity of angle $(n+2)\pi$ (here n is the order of the zero). At the poles, a neighborhood either has a "fold" (for a simple pole), or is isometric to a half-infinite cylinder (for a pole of order two) or has the structure of a *planar end* defined below (for higher-order poles). See [Str84], and [Gupb] for a discussion.

In what follows we think of a Euclidean half-plane as the region on one side of the vertical (y-) axis on the plane, which we identify with $\mathbb{R}$.

Definition 2.1 (Planar-end, metric residue)**.** Let $\{H_i\}$ for $1 \leq i \leq n$ be a cyclically ordered collection of half-planes with rectangular "notches" obtained by deleting, from each, a rectangle of horizontal and vertical sides adjoining the boundary, with the boundary segment having end-points a_i and b_i, where $a_i < b_i$. A *planar end* is obtained by gluing the interval $[b_i, \infty)$ on ∂H_i with $(-\infty, a_{i+1}]$ on H_{i+1} by an orientation-reversing isometry. Such a surface is homeomorphic to a punctured disk, and has a *metric residue* defined to be absolute value of the alternating sum $\sum\limits_{i=1}^{n}(-1)^{i+1}(b_i - a_i)$.

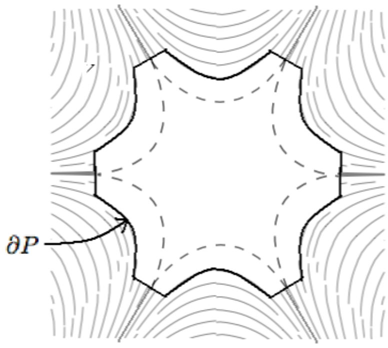


FIGURE 1. A planar end P involving six half-planes with rectangular "notches", shown with its vertical foliation and polygonal boundary.

Remark. A planar end of order n and metric residue a is isometric to a planar domain containing 0, equipped with the meromorphic quadratic differential $q_0 = \left(\frac{1}{z^{n+2}} + \frac{ia}{z^{n/2+2}}\right)dz^2$. (See §3.3 of [Gupb].)

Measured foliations. A holomorphic quadratic differential $q \in Q(X)$ also determines a *horizontal foliation* on X which we denote by $\mathcal{F}_h(q)$, obtained by integrating the line field of vectors $\pm v$ where the quadratic differential is real and positive, that is $q(v,v) \geq 0$. Similarly, there is a *vertical foliation* $\mathcal{F}_v(q)$ consisting of integral curves of directions where q is real and negative.

Note that we can also talk of horizontal or vertical *segments* on the surface, as well as horizontal and vertical *lengths.*

The foliations above are *measured*: the measure of an arc transverse to $\mathcal{F}_h$ is given by its *vertical* length, and the transverse measure for $\mathcal{F}_v$ is given by horizontal lengths. Such a measure is invariant by isotopy of the arc if it remains transverse with endpoints on leaves.

Let $\mathcal{MF}$ be the space of such measured foliations upto isotopy and Whitehead-equivalence (see [FLP79]). For any fixed X, any $\mathcal{F} \in \mathcal{MF}$ is the vertical foliation for a unique $q \in \mathcal{Q}(X)$

([HM79]). Moreover, a choice of transverse measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ determines a unique X and $q \in \mathcal{Q}(X)$ such that $\mathcal{V}(q) = \mathcal{F}_1$ and $\mathcal{H}(q) = \mathcal{F}_2$ where the equalities are as elements in $\mathcal{MF}$.

2.3. **Geodesic laminations.** A *geodesic lamination* on a hyperbolic surface is a closed subset of the surface which is a union of disjoint simple geodesics. Any geodesic lamination λ is a disjoint union of sublaminations

$$\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \lambda_N \cup \gamma_1 \cup \gamma_2 \cdots \gamma_M \tag{1}$$

where λ_is are minimal components (with each half-leaf dense in the component) which consist of uncountably many geodesics (a Cantor set cross-section) and the γ_js are isolated geodesics (which in particular could be "spiralling", see [CB88]).

A *measured* geodesic lamination is equipped with a transverse measure μ, that is a measure on arcs transverse to the lamination which is invariant under sliding along the leaves of the lamination. It can be shown that for the support of a measured lamination the isolated leaves in (1) above are weighted simple closed curves (ruling out the possibility of isolated geodesics spiralling onto a closed component). We call a lamination *arational* if it consists of a single minimal component that is *maximal*, that is, whose complementary regions are ideal hyperbolic triangles.

Remark. Sometimes in the literature "arational" denotes a minimal lamination with *polygonal* complementary regions. We use "arational" in our case however, to avoid the more awkward description of the lamination "being maximal and minimal".

Any measured geodesic lamination corresponds to a unique measured foliation of the surface, obtained by 'collapsing' the complementary components. Conversely, any measured foliation can be 'tightened' to a geodesic lamination, and hence the two are equivalent notions (see, for example, [Lev83] or [Kap09]).

2.4. **Train tracks.** A *train-track* on a surface is an embedded C^1 graph with a labelling of incoming and outgoing half-edges at every vertex (*switch*). A *weighted* train-track comes with an assignment of non-negative real numbers (*weights*) to the edges (*branches*) such that at every switch, the sums of the weights of the incoming and outgoing branches are equal. A standard reference is [PH92]. The leaves of a measured lamination (or a sufficiently long simple closed curve) lie close to such a train-track, and the transverse measures provide the weights. This provides a convenient combinatorial encoding of a lamination (see, for example, [FLP79] or [Thu82]).

2.5. **Teichmüller rays.** A geodesic ray in the Teichmüller metric (defined in §2.1) starting from a point X in $\mathcal{T}_g$ and in a direction determined by a holomorphic quadratic differential $\phi \in Q(X)$ (or equivalently by a lamination $\lambda \in \mathcal{ML}$) is obtained by starting with the singular flat surface X and scaling the transverse measure of λ by a factor of e^{2t}. Alternatively, it is the new singular-flat surface determined by the pair of transverse measured foliations $(e^{2t}\mathcal{F}_v, \mathcal{F}_h)$.

Note that we shall use the convention that the lamination λ is the *vertical* foliation, so the stretching of the metric described above is transverse to it.

Remark. By a conformal rescaling, this is equivalent to the usual definition involving scaling the horizontal direction by a factor e^t and the vertical direction by a factor e^{-t}. This has the advantage of preserving the q-area, however in our case we shall be considering certain geometric limits of infinite area.

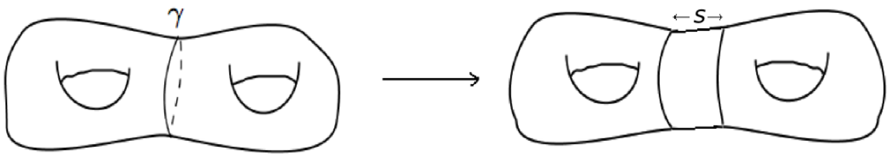


FIGURE 2. The grafting map along the weighted curve $s\gamma$.

2.6. **Grafting.** The *conformal* grafting map

$$gr : \mathcal{T}_g \times \mathcal{ML} \to \mathcal{T}_g$$

is defined as follows: for a simple closed curve γ with weight s, grafting a hyperbolic surface X along $s\gamma$ inserts a Euclidean annulus of width s along the geodesic representative of γ (see Figure 2). The resulting surface acquires a $C^{1,1}$-metric (see [SW02] and also [KP94]) and a new conformal structure which is the image $gr(X, s\gamma)$. Grafting for a *general* measured lamination λ is defined by taking the limit of a sequence of approximations of λ by weighted simple closed curves, that is, a sequence $s_i\gamma_i \to \lambda$ in $\mathcal{ML}$.

Notation. We often write $gr(X, \lambda)$ as $gr_\lambda(X)$.

The hybrid Euclidean-and-hyperbolic metric on the grafted surface is called the *Thurston metric*. The length in the Thurston metric on $gr_\lambda X$ of an arc τ intersecting λ, is its hyperbolic length on X plus its transverse measure. We shall refer to the *Euclidean* length (of any arc) as its transverse measure (total length minus the hyperbolic length).

For further details, as well as the connection with complex projective structures, see [Dum] for an excellent survey.

It is known that for any fixed lamination λ, the grafting map $(X \mapsto gr_\lambda X)$ is a self-homeomorphism of $\mathcal{T}_g$ ([SW02]).

Definition 2.2. A *grafting ray* from a point X in $\mathcal{T}_g$ in the 'direction' determined by a lamination λ is the 1-parameter family $\{X_t\}_{t\geq 0}$ where $X_t = gr_{t\lambda}(X)$.

3. GEOMETRY OF GRAFTED SURFACES

The surface obtained by grafting a hyperbolic surface X along a measured lamination λ acquires a conformal metric (the *Thurston metric*) that is Euclidean in the grafted region and hyperbolic elsewhere - see the preceding description in §2.6. The case of λ a weighted simple closed curve is easily described: the metric comprises a Euclidean cylinder of length equal to the weight, inserted at the geodesic representative of λ (see Figure 2). A general measured lamination has more complicated structure, with uncountably many geodesics winding around the surface. In this section we briefly recall some of the previous work in [Gupa] that develops an understanding of the underlying conformal structure of the grafted surface when one grafts along such a lamination.

Though the finer structure of a general measured lamination is complicated, it can be combinatorially encoded by a "train-track neighborhood" on the surface which also allows a convenient description of the grafted metric. For small $\delta > 0$, the train-track neighborhood $\mathcal{T}_\delta$ of a measured lamination λ is a Hausdorff neighborhood of maximum width δ, that can be thought of as a thickening of an embedded train-track graph (§2.4) - the leaves of the λ run along embedded rectangles that collapse onto its branches, and the branch weights are the transverse measures of arcs across them.

The intuition is that as one grafts, the subsurface $\mathcal{T}_\delta$ widens in the transverse direction (along the "ties" of the train-track neighborhood), and conformally approaches a union of wide Euclidean rectangles. The complement $X \setminus \lambda$ is unaffected by grafting: it may consist of ideal polygons or subsurfaces with moduli.

3.1. **Train-track decomposition.** We summarize the construction (see §4.1 of [Gupa] for details) of the afore-mentioned subsurface $\mathcal{T}_\delta \subset X$ containing the lamination λ. The parameter δ in the construction shall be sufficiently small, as determined later in Lemma 6.13 in §6.5.

Recall that λ has a decomposition (1) into minimal components and simple closed curves.

Each minimal component of λ has a support equipped with a transverse *horocyclic foliation* $\mathcal{H}$ obtained by considering the horocyclic foliation in the "spikes" of the complementary cusped hyperbolic surfaces (see §3.1 of [Gupa]). For that minimal component of λ, choose a transverse arc τ that is a segment of $\mathcal{H}$ of hyperbolic length sufficiently small, depending on δ (see Lemma 3.1) and use the first return map of τ to itself (following the leaves of λ) to form a collection of rectangles $R_1, R_2, \ldots R_N$ with vertical geodesic sides, and horizontal sides lying on τ.

For each simple closed geodesic component, consider an annulus of hyperbolic width 2δ containing it as a central core curve.

We define

$$\mathcal{T}_\epsilon = R_1 \cup R_2 \cup \cdots R_N \cup A_1 \cup \cdots A_M \tag{2}$$

to be the union of these rectangles and annuli on the surface X. This contains (or *carries*) the lamination λ.

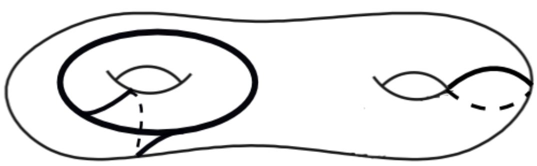


FIGURE 3. The train track $\mathcal{T}_\delta$ is a union of rectangular and annular branches - see (2). These contain the lamination, and widen as one grafts.

Lemma 3.1 (Lemma 4.2 in [Gupa]). *For any sufficiently small $\delta > 0$, if the hyperbolic length of τ is sufficiently small, the height of each of the rectangles $R_1, R_2, \ldots R_N$ is greater than $\frac{1}{\delta^2}$, and the hyperbolic width is less than δ.*

Remark. Henceforth we shall assume that δ is chosen smaller than ϵ (how sufficiently small shall be ascertained later, in Lemma 6.13).
The complement of $\mathcal{T}_\delta$ is a union of subsurfaces $T_1, T_2, \ldots T_k$.

Thickening. We also describe a "thickening" of $\mathcal{T}_\delta$ (or "trimming" of the above subsurfaces) to ensure that λ is contained *properly* in $\mathcal{T}_\delta$:

For each geodesic side in $T_1, T_2, ...T_k$, choose an adjacent thin strip (inside the region in the complement of $\mathcal{T}_\delta$) and bounded by another geodesic segment "parallel" to the sides, and append it to the adjacent rectangle of the train-track.

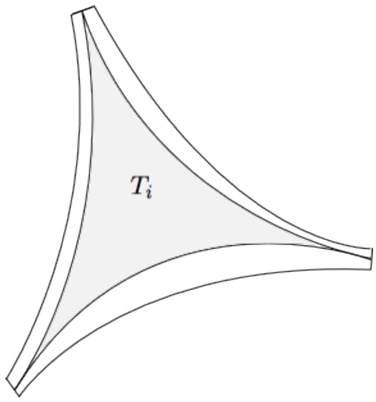


FIGURE 4. Rectangular strips in the complementary regions are appended to the branches of $\mathcal{T}_\delta$.

We continue to denote the collection of this slightly thickened rectangles by $R_1, R_2, \ldots R_N$, and their union by $\mathcal{T}_\delta$.

Along the grafting ray determined by (X, λ), the rectangles get wider, and one gets a decomposition of each grafted surface $gr_{t\lambda}X$ into rectangles and annuli and these complementary regions, with the same combinatorics of the gluing.

As in [Gupa], the *total width* of a rectangle in the above decomposition is the maximum width in the Thurston metric, and its *Euclidean width* is its transverse measure. The following fact is a consequence of the definition of grafting:

Lemma 3.2 (See §4.1.3 of [Gupa]). *If P is a rectangle or annulus in the above decomposition, its Euclidean width on $gr_{t\lambda}X$ is $t\mu(P)$ where $\mu(P)$ is its transverse measure on X.*

3.2. **Almost-conformal maps.** Recall that by the definition of the Teichmüller metric in §2.1, if there exists a map $f : X \to Y$ which is $(1 + \epsilon)$-quasiconformal, then

$$d_{\mathcal{T}}(X, Y) = O(\epsilon).$$

Notation. Here, and throughout this article, $O(\alpha)$ refers to a quantity bounded above by $C\alpha$ where $C > 0$ is some constant depending only on genus g (which remains fixed), the exact value of which can be determined *a posteriori*.

As in [Gupa], we shall use:

Definition 3.3. An *almost-conformal map* shall refer to a map which is $(1 + O(\epsilon))$-quasiconformal.

We shall be constructing such almost-conformal maps between surfaces in our proof of the asymptoticity result (Theorem 1.1) and using the notions in this section.

In §4.2 and §6.1 of [Gupa] we developed the notion of "almost-isometries" and an "almost-conformal extension lemma" that we restate (we refer to the previous paper for the proof).

Definition 3.4. A homeomorphism f between two C^1-arcs on a surface with a C^1-metric is an (ϵ, A)-*almost-isometry* if f is continuously differentiable with dilatation d (the supremum of the derivative of f over the domain arc) that satisfies $|d-1| \leq \epsilon$ and such that the lengths of any subinterval and its image differ by an additive error of at most A.

Definition 3.5. A map f between two rectangles is (ϵ, A)-*good* if it is isometric on the vertical sides and (ϵ, A)-almost-isometric on the horizontal sides.

Remark. A "rectangle" in the above definition refers to the four boundary arcs of an embedded quadrangular region on a surface with a conformal metric, that intersect at right angles, with a pair of opposite sides being identified as *vertical* and the other pair called *horizontal*.

The following lemma from [Gupa] (Lemma 6.8 in that paper) shall be used in the final construction in §7.5:

Lemma 3.6 (Qc extension)**.** *Let R_1 and R_2 be two Euclidean rectangles with vertical sides of length h and horizontal sides of lengths l_1 and l_2 respectively, such that $l_1, l_2 > h$ and $|l_1 - l_2| < A$, where $A/h \leq \epsilon$. Then any (ϵ, A)-good map $f : \partial R_1 \to \partial R_2$ has a $(1 + C\epsilon)$-quasiconformal extension $F : R_1 \to R_2$ for some (universal) constant $C > 0$.*

3.3. **Model rectangles.** Recall that the train-track decomposition of X persists as we graft along λ, with the Euclidean width of the rectangles $R_1, R_2, \ldots, R_N$ and annuli $A_1, A_2 \ldots A_M$ increasing along the λ-grafting ray. We denote the grafted rectangles and annuli on $gr_{t\lambda}X$ by $R_t^1, R_t^2, \ldots R_t^N, A_t^1, A_t^2, \ldots A_t^M$.

The annuli are purely Euclidean, but the rectangles have (typically infinitely many) hyperbolic strips through them. Their Euclidean width is proportional to the transverse measure (Lemma 3.2). The *total width* $w_i(t)$ of the rectangle R_t^i is the maximum width in the Thurston metric on $gr_{t\lambda}X$, and we have

$$|w_i(t) - tw_i| < \epsilon \tag{3}$$

since the initial *hyperbolic* widths of the rectangles on X is less than ϵ by the remark following Lemma 3.1.

Though having a hybrid hyperbolic-and-Euclidean metric, a rectangular piece R^t from the collection $\{R_t^1, R_t^2, \ldots, R_t^N\}$ has an almost-conformal *Euclidean model rectangle*, as proved in §4.3 of [Gupa] (see also Lemma 6.9 in that paper). We restate that result, and since it is crucial in our constructions, we provide a sketch of the proof:

Lemma 3.7 (Lemma 6.19 of [Gupa])**.** *Let R be a rectangle of width w from the collection $\{R^1, R^2, \ldots, R^N\}$ of the decomposition of X. For any $t > 1$, there is a $(1 + C\epsilon)$-quasiconformal map from R_t to a Euclidean rectangle of width tw which is (ϵ, ϵ)-good on the boundary. (Here $C > 0$ is some universal constant.)*

Sketch of the proof. One can approximate the measured lamination $\lambda \cap R$ passing through the rectangle (on the ungrafted surface) by *finitely* many weighted arcs by a standard finite-approximation of the measure. Grafting R along this finite approximation λ_i amounts to splicing in Euclidean strips at the arcs, of width equal to their weights. The hyperbolic rectangle R is ϵ-thin (put $t = 0$ in (3)).
Working in the upper-half-plane model of the hyperbolic plane, one can map R to the Euclidean plane by an almost-conformal map that "straightens" the horocyclic foliation

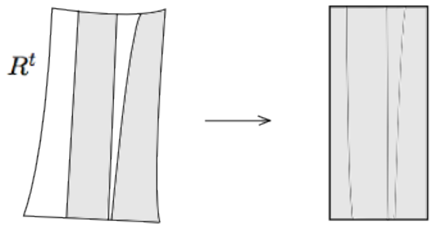


FIGURE 5. The thin hyperbolic pieces (shown unshaded) of the grafted rectangle can be mapped to the Euclidean plane and together with a map of the Euclidean pieces defines an almost conformal map to a rectangle (Lemma 3.7).

across it. The images of the vertical sides are "almost-vertical" and the finitely many Euclidean strips that need to be spliced in can be mapped in by almost-conformal maps such that the images fit together to a form an "almost" Euclidean rectangle (with almost-vertical sides). The condition that $t > 1$ implies that this can finally be adjusted to a Euclidean rectangle by a horizontal stretch with stretch-factors ϵ-close to 1 at each height. Call this composite almost-conformal map f_i. One then takes a limit of these f_i-s for a sequence of approximations $\lambda_i \to \lambda \cap R$. □

3.4. **Limits of grafting rays.** Consider a grafting ray $\{X_t\}_{t\geq 0}$ determined by a hyperbolic surface X and a geodesic lamination λ. In this section we shall introduce the "limiting surface" along such a ray.

Definition 3.8 (X_∞)**.** Consider the metric completion $\widehat{X \setminus \lambda}$ of the hyperbolic subsurface in the complement of the lamination. Each boundary component of this completion is either closed (topologically a circle) or "polygonal", comprising a closed chain of bi-infinite hyperbolic geodesics that form "spikes" (see Figure 6.) Construct X_∞ by gluing in Euclidean half-infinite cylinders along the geodesic boundary circles, and Euclidean half-planes along the geodesic boundary lines, and where the gluings are by isometries along the boundary. The resulting (possibly disconnected) surface X_∞ acquires a conformal structure, and a C^1-metric that is a hybrid of Euclidean and hyperbolic metrics.

Example. Before the general construction, consider the case when the lamination is a single simple closed geodesic γ, The grafting ray X_t comprises longer Euclidean cylinders grafted in at γ. The conformal limit X_∞ in this case is the hyperbolic surface $X \setminus \gamma$ with half-infinite Euclidean cylinders glued in at the boundary components.

Let

$$\widehat{X \setminus \lambda} = S^1 \sqcup S^2 \cdots \sqcup S^k \tag{4}$$

be the decomposition into connected components, each of which get a marking induced from that on X. From the above construction we have:

$$X_\infty = S^1_\infty \sqcup \cdots \sqcup S^k_\infty \tag{5}$$

where S^j_∞ is obtained by attaching half-planes and half-cylinders to the boundary of S^j, for each $1 \leq j \leq k$. Moreover, each S^j_∞ acquires a marking from that on S^j.

For later use, we note the following property of these limit surfaces:

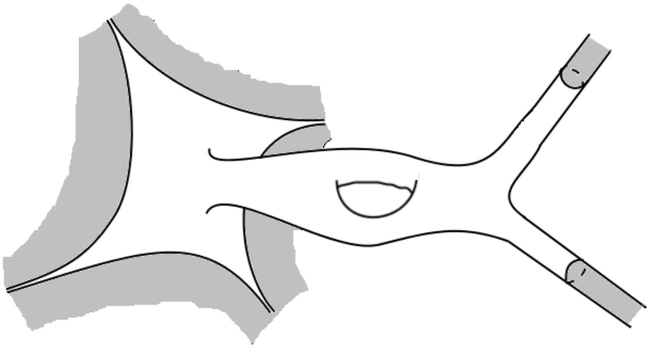


FIGURE 6. An example of the surface X_∞. The shaded regions are half-planes and half-infinite cylinders.

Lemma 3.9. *For any $\epsilon > 0$, there is a $T > 0$ such that for each $t > T$, the surface X_t along the grafting ray admits a decomposition:*

$$X_t = S_t^1 \sqcup S_t^2 \sqcup \cdots \sqcup S_t^k \tag{6}$$

such that there is a $(1+\epsilon)$-quasiconformal embedding $e^j : S_t^j \to S_\infty^j$ that preserves markings, for each $1 \leq j \leq k$.

Proof. Fix an $\epsilon > 0$. Recall the train-track decomposition into rectangles and annuli that carry the lamination (see §3.1). As before, label the components of the complementary subsurface $X \setminus \mathcal{T}_\delta$ by $T_1, T_2, \ldots T_k$. Each T_i is a hyperbolic surface with boundary components either closed curves, or having geodesic segments that bound truncated "spikes" (for example, a truncated ideal triangle).

We define a "polygonal piece" S_t^j by expanding each T_i for $1 \leq i \leq k$: each geodesic side of ∂T_i has an adjacent rectangle, and each closed circle has an adjacent annulus, and we append exactly half of those adjacent pieces to T_i. These "expanded" pieces now cover the entire surface X_t. This defines a decomposition into subsurfaces as in (6). Each rectangular piece has a hybrid Euclidean-and-hyperbolic C^1-metric on it (see the left figure in Figure 5).

Moreover, recall from Lemma 3.2 that the Euclidean widths of the rectangles and annuli increase linearly in t. The grafted annuli are entirely Euclidean, and for sufficiently large t, the rectangular pieces admit $(1+\epsilon)$-quasiconformal maps to Euclidean rectangles (Lemma 3.7). These almost-conformal maps together with an isometry on T_i, define a a $(1+\epsilon)$-quasiconformal embedding of the subsurface S_t^j admits to S_∞^j. □

3.5. **Asymptoticity in the arational case.** We briefly summarize the strategy of the proof of Theorem 1.1 in the case when the lamination λ is arational. This was carried out in [Gupa], and we refer to that paper for details.

In this case the train-track $\mathcal{T}_\delta$ carrying λ is maximal, that is, its complement is a collection of truncated ideal triangles, and one can build the *horocyclic foliation* $\mathcal{H}$ transverse to the lamination. The branches widen along a grafting ray as more Euclidean region is grafted in (Lemma 3.2), and $\mathcal{H}$ lengthens. One can define a singular flat surface $\hat{X}_t$ obtained by collapsing the hyperbolic part of the Thurston metric on X_t along $\mathcal{H}$, and it is not hard to check that these lie along a common Teichmüller ray determined by a measured foliation equivalent to λ.

The maps of Lemma 3.7 from the rectangles on the grafted surface to Euclidean rectangles piece together to form a quasiconformal map from X_t to $\hat{X}_t$ that is almost-conformal for *most* of the grafted surface (Lemma 4.22 of [Gupa]). For all sufficiently large t, this can be adjusted to an almost-conformal map of the entire surface by a version of Corollary 4.7 in the next section. Since the singular flat surfaces $\hat{X}_t$ lie along a common Teichmüller ray, as $\epsilon \to 0$ this proves Theorem 1.1 for arational laminations.

3.6. **Asymptoticity in the multicurve case.** We briefly recall the strategy for the case when λ is a multicurve which, along with the arational case was dealt with in [Gupa]. As described in §3.4 along the grafting ray the surfaces acquire increasingly long Euclidean cylinders along the geodesic representatives of the curves, and one considers the conformal limit X_∞ that has half-infinite Euclidean cylinders inserted at the boundary components of $X \setminus \lambda$.

A theorem of Strebel then shows the existence of a certain meromorphic quadratic differential with poles of order two on X_∞. This produces a singular flat surface Y_∞ comprising half-infinite Euclidean cylinders, together with a conformal map $g : X_\infty \to Y_\infty$. Suitably truncating the cylinders on X_∞ and Y_∞, adjusting g to an almost-conformal map between them, and gluing the truncations, produces for all sufficiently large t, an almost-conformal map between X_t and a surface along a Teichmüller ray $\{Y_t\}$ that limits to Y_∞. Details are in §5 of [Gupa].

In the more general case handled in this paper, one needs Theorem A.7 (see the Appendix), which is the appropriate generalization of the theorem of Strebel mentioned above.

4. A quasiconformal toolkit

In this section we collect some constructions and extensions of quasiconformal maps that shall be useful in the proof of Theorem 1.1. This forms the technical core of this paper, and the reader is advised to skip it at first reading, and refer to the lemmas whenever they are used later.

Most of the results here are probably well-known to experts, however in our setting we need care to maintain almost-conformality of the maps (see §3.2). For a glossary of known results we refer to the Appendix of [Gupa].

Throughout, $\mathbb{D}$ shall denote the unit closed disk on the complex plane, and B_r shall denote the closed disk of radius r centered at 0. Note that any quasiconformal map defined on the interior of a Jordan domain extends to a homeomorphism of the boundary (see, for example, [AB56]).

4.1. **Interpolating maps.** We start with the following observation about the Ahlfors-Beurling extension that was used in a construction in [AJKS10]:

Lemma 4.1 (Interpolating with identity). *Let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be a $(1+\epsilon)$-quasisymmetric map. Then there exists an $0 < s < 1$ and a homeomorphism $H : \mathbb{D} \to \mathbb{D}$ such that*
(1) H is $(1 + C\epsilon)$-quasiconformal.
(2) $H|_{\partial\mathbb{D}} = h|_{\partial\mathbb{D}}$.
(3) H restricts to the identity map on B_s.
Here $C > 0$ is a universal constant, and s above depends only on ϵ.

Proof. As in §2.4 of [AJKS10], lift h to a homeomorphism $\tilde{h} : \mathbb{R} \to \mathbb{R}$ that satisfies $\tilde{h}(x+1) = \tilde{h}(x) + 1$, and consider the Ahlfors-Beurling extension of $\tilde{h}$ to the upper half plane $\mathbb{H}$:

$$F(x+iy) = \frac{1}{2}\int\limits_0^1 \tilde{h}(x+ty) + \tilde{h}(x-ty)dt + i\int\limits_0^1 \tilde{h}(x+ty) - \tilde{h}(x-ty)dt$$

It follows from the periodicity that

$$F(x+i) = x + i + c_0$$

where $c_0 = \int\limits_0^1 \tilde{h}(t)dt - 1/2 \in [-1/2, 1/2]$.

We note that since we have used a locally conformal change of coordinates $w \mapsto e^{2\pi i w}$ between $\mathbb{H}$ and $\mathbb{D}$, we have that $\tilde{h}$ is also $(1+O(\epsilon))$-quasisymmetric, and F is almost-conformal.

For $D > 1$ we can define a map $F_1 : \mathbb{H} \to \mathbb{H}$ which restricts to F on $\mathbb{R} \times [0,1]$, and the identity map for $y \geq D$ and interpolates linearly on the strip $\mathbb{R} \times [1, D]$:

$$F_1(x+iy) = x + iy + c_0\left(\frac{D-y}{D-1}\right)$$

For D sufficiently large (greater than $1/C\epsilon$), F_1 is almost-conformal everywhere as can be checked by computing derivatives on the interpolating strip. Since $F_1(z+k) = F_1(z) + k$ for all $z \in \mathbb{H}$ and $k \in \mathbb{Z}$, it descends to an almost-conformal map $H : \mathbb{D} \to \mathbb{D}$ that restricts to h on $\partial\mathbb{D}$ and the identity map on B_s for $s = e^{-2\pi D}$. □

Notation. In the statements of the following lemmas, we use the same C to denote universal constants that might *a priori* vary between the lemmas, since one can, if needed, take a maximum of them and fix a single constant that works for each.

The following corollary of the above lemma interpolates a quasiconformal map with the identity map on the *outer* boundary:

Lemma 4.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a $(1+\epsilon)$-quasiconformal map such that $f(0) = 0$. Then there exists an $0 < r_0 < 1$ and a map $F : \mathbb{D} \to \mathbb{D}$ such that*
(1) F is $(1 + C\epsilon)$-quasiconformal.
(2) $F|_{B_{r_0}} = f|_{B_{r_0}}$.
(3) $F|_{\partial\mathbb{D}}$ is the identity.
Here $C > 0$ is a universal constant.

Proof. Since f is almost-conformal, so is f^{-1}, and the latter extends to the boundary and restricts to a homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}$ that is $(1+O(\epsilon))$-quasisymmetric. By Lemma 4.1 there exists an almost-conformal extension $H : \mathbb{D} \to \mathbb{D}$ of h that restricts to the identity on B_{r_0} for sufficiently small r_0. The composition $H \circ f$ then is the required map F that restricts to f on B_{r_0} and is identity on $\partial\mathbb{D}$. □

We shall now obtain an interpolation of an almost-conformal map with a *given* conformal map at the outer boundary. We start with the following observation:

Lemma 4.3. *Let $\epsilon, c > 0$. Then for all sufficiently small $0 < r < \min\{1, c\}$, there is a $(1+\epsilon)$-quasiconformal map $f : \mathbb{D} \to B_c$ such that f restricts to the identity map on B_r and to the dilatation $z \mapsto cz$ on $\partial\mathbb{D}$.*

Proof. For any $0 < r < \min\{1, c\}$, consider the map $f_r : \mathbb{D} \to \mathbb{D}$ defined by:

$$f_r(z) = rz/|z| + z(c - r)$$

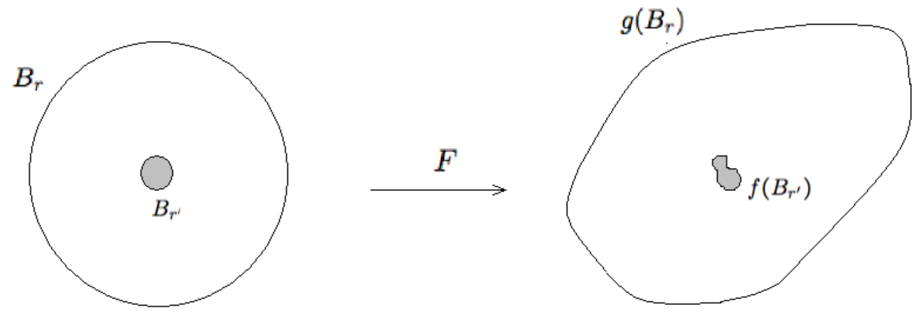


FIGURE 7. In Lemma 4.5 we interpolate from the map g on ∂B_r to the almost-conformal map f between the shaded regions.

on $\mathbb{D} \setminus B_r$, and $f_r(z) = z$ on B_r.
Note that this map restricts to the dilatation $z \mapsto cz$ on $|z| = 1$ as required. Computing partial derivatives, we obtain:

$$\partial_{\bar{z}} f_r(z) = -\frac{rz^2}{2|z|^3}$$

and

$$\partial_z f_r(z) = \frac{r}{|z|} - \frac{r|z|}{2} + (c - r)$$

and from this it is easy to check that the quasiconformal dilatation:

$$K(f_r) = \frac{|\partial_{\bar{z}} f_r| + |\partial_z f_r|}{|\partial_{\bar{z}} f_r| - |\partial_z f_r|} \to 1$$

as $r \to 0$.
Hence there exists a (sufficiently small) r_0 such that if $r < r_0$ then the function f_r above satisfies $K(f_r) \leq 1 + \epsilon$, as required. □

This yields the following strengthening of Lemma 3.8 of [Gupc] (see also Lemma 5.1 of [Gupa]):

Lemma 4.4. *Let $g : \mathbb{D} \to \mathbb{C}$ be a univalent conformal map such that $g(0) = 0$. Then for any $\epsilon > 0$ there is a (sufficiently small) $r > 0$ and a $(1 + \epsilon)$-quasiconformal map $F : \mathbb{D} \to g(\mathbb{D})$ that restricts to the identity map on B_r and agrees with g on $\mathbb{D} \setminus B_{2r}$.*

Note. Lemma 3.8 of [Gupc] asserts that under the same hypotheses, if $c = g'(0)$, then there is a $(1 + \epsilon)$-quasiconformal map $f : \mathbb{D} \to g(\mathbb{D})$ that restricts to the dilatation $z \mapsto cz$ on B_r and agrees with g on $\mathbb{D} \setminus B_{2r}$, for some sufficiently small r.

Proof. Let $c = g'(0)$. As a first step, we apply Lemma 3.8 of [Gupc] mentioned above to get an f_1 restricting to the dilatation $z \mapsto cz$ on B_s for some sufficiently small s. We can now apply a (suitably scaled) version of Lemma 4.3 to obtain a $(1 + \epsilon)$-quasiconformal map $f : B_s \to B_s$ that restricts to the dilatation $z \mapsto cz$ on ∂B_s and to the identity map on B_r, for some sufficiently small r. Together with f_1 on $\mathbb{D} \setminus B_s$, this yields a map F as required. □

Remark. By conjugating by the dilation $z \mapsto (1/r)z$ the above result holds (for some $0 < s < r$) if the conformal map g is defined only on $B_r \subset \mathbb{D}$.

Lemma 4.5 (Interpolation)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a $(1 + \epsilon)$-quasiconformal map such that $f(0) = 0$, and let $g : B_r \to g(B_r) \subset \mathbb{D}$, for some $0 < r < 1$, be a conformal map such that $g(0) = 0$. Then there exists an $0 < r' < r < 1$ and a map $F : B_r \to g(B_r)$ such that*
(1) $F|_{B_{r'}} = f|_{B_{r'}}$.

(2) $F|_{\partial B_r} = g|_{\partial B_r}$.
(3) F *is* $(1 + C\epsilon)$*-quasiconformal, for some universal constant* $C > 0$.

Proof. By Lemma 4.4 (see also the following remark) there exists an $0 < s < r < 1$ and an almost-conformal map $G : B_r \to g(B_r)$ that restricts to g on ∂B_r and is identity on B_s. Now we can apply Lemma 4.1 to the rescaled map $f_s = (1/s)f|_{B_s} : \mathbb{D} \to \mathbb{D}$ to get a map $F_s : \mathbb{D} \to \mathbb{D}$ that restricts to f_s on some $B_{s'}$ for $0 < s' < 1$ and the identity map on the boundary $\partial\mathbb{D}$. Rescaling back, we get a map $F_1 : B_s \to B_s$ that restricts to the identity map on ∂B_s and to f on $B_{r'}$ where $r' = s's < s$. Since F_1 and G are both identity on ∂B_s, F_1 together with the restriction of G on $B_r \setminus B_s$ defines the required interpolation $F : B_r \to g(B_r)$. □

An extension lemma. The following lemma has been proved in [Gupb] (for a weaker version see Lemma A.1 of [Gupa]).

Lemma 4.6. *For any* $\epsilon > 0$ *sufficiently small, and* $0 \leq r \leq \epsilon$, *a map*

$$f : \mathbb{D} \setminus B_r \to \mathbb{D}$$

that
(1) preserves the boundary $\partial\mathbb{D}$ *and is a homeomorphism onto its image,*
(2) is $(1 + \epsilon)$*-quasiconformal on* $\mathbb{D} \setminus B_r$
extends to a $(1 + C\epsilon)$*-quasisymmetric map on the boundary, where* $C > 0$ *is a universal constant.*

Remark. The boundary correspondence for disks (the case when $r = 0$) is proved in [AB56], and we employ their methods in the proof.

The following corollary also follows from Theorem 2.7 of [Mon09]:

Corollary 4.7. *Let* $\epsilon > 0$ *be sufficiently small, and* U_0, U *and* U' *be embedded disks on a Riemann surface such that* $U_0 \subset U$ *and the annulus* $A = U \setminus U_0$ *has modulus larger than* $\frac{1}{2\pi} \ln \frac{1}{\epsilon}$. *Then for any conformal embedding* $g : A \to U'$ *there is a* $(1 + C'\epsilon)$*-quasiconformal map* $f : U \to U'$ *such that* f *and* g *are identical on* ∂U. *(Here* $C' > 0$ *is a universal constant.)*

Proof. By uniformizing, one can assume that $U = U' = \mathbb{D}$ and $U_0 \subset B_r$ where $r \leq \epsilon$ by the condition on modulus. Now we are in the setting of the previous lemma, since g being conformal is also $(1 + \epsilon)$-quasiconformal on A. One can hence conclude that g extends to a $(1 + C\epsilon)$-quasisymmetric map of the boundary, which by the Ahlfors-Beurling extension (see [AB56]) extends to an $(1 + C'\epsilon)$-quasiconformal map of the entire disk, which is our required map f. □

5. Strategy of the proof

The proof of Theorem 1.1 following the outline in this section is carried out in §6.

As before we fix a hyperbolic surface X and measured lamination λ and consider the grafting ray $X_t = gr_{t\lambda}X$. Our task is to find a Teichmüller ray Y_t such that under appropriate parametrization, the Teichmüller distance between the rays tends to zero.

The strategy of the proof is a generalization of the argument when λ is a multicurve (see §3.6).

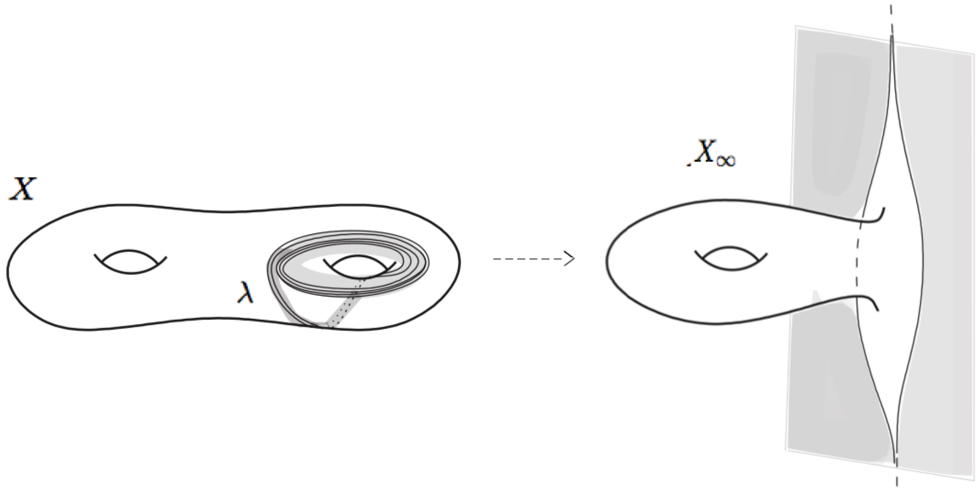


FIGURE 8. An example of the conformal limit X_∞ for a minimal lamination λ, obtained by attaching Euclidean half-planes along the boundary of the completion of $X \setminus \lambda$.

Proof outline. Fix an $\epsilon > 0$. The proof of Theorem 1.1 will be complete if one can show that for all sufficiently large t, there exists a $(1+\epsilon)$-quasiconformal map:

$$f : X_t \to Y_t$$

where X_t is the surface along the grafting ray determined by (X, λ), and Y_t lies along a Teichmüller ray determined by λ.

Recall from §3.4 that there is a limit of the grafting ray with the decomposition:

$$X_\infty = S^1_\infty \sqcup S^2_\infty \sqcup \cdots \sqcup S^k_\infty \tag{7}$$

where each S^j_∞ is a surface obtained by appending half-planes and half-infinite cylinders to the boundary of a connected component of $\widehat{X \setminus \lambda}$.

Our strategy is outlined as follows:

Step 1. By specifying an appropriate meromorphic quadratic differential on each $S^1_\infty, \ldots S^k_\infty$ using Theorem A.7, we find a singular flat surface

$$Y_\infty = Y^1_\infty \sqcup Y^2_\infty \sqcup \cdots \sqcup Y^k_\infty$$

conformally equivalent to X_∞ (the singular flat metric is the one induced by the differential). The "local data" (eg. orders and residues) at the poles of the meromorphic quadratic differential are prescribed according to the geometry of the "ends" of each S^j_∞. The fact that the underlying marked Riemann surfaces are identical then gives conformal homeomorphisms $g^j : S^j_\infty \to Y^j_\infty$ each in the correct isotopy class.

Step 2. By the quasiconformal interpolation of Lemma 4.5, each conformal map g^j of *Step 1* is adjusted to produce an $(1+\epsilon)$-quasiconformal map $h^j : S^j_\infty \to Y^j_\infty$ that is "almost the identity map" near the ends. In particular, for any "truncation" of those infinite-area surfaces at sufficiently large height, the map h^j is chosen to preserve the horizontal and vertical directions, and be almost-isometric on the resulting polygonal boundaries.

Step 3. By Lemma 3.9 the grafted surface at time t has the decomposition:

$$X_t = S_t^1 \sqcup S_t^2 \sqcup \cdots \sqcup S_t^k$$

where each S_t^j is obtained by appending halves of adjacent branches of a train-track neighborhood $\mathcal{T}_\delta$ carrying the lamination. The Euclidean part of these branches can be glued up to produce singular flat surfaces $Y_t^1, \ldots Y_t^k$ that embed isometrically in $Y_\infty^1, \ldots Y_\infty^k$ respectively. Gluing these in the pattern determined by that of the S_t^j-s then produces a surface Y_t which we show lies on a common Teichmüller ray (in a direction given by λ).

Step 4. By Lemma 3.9, for sufficiently large t the surfaces $S_t^1, S_t^2, \ldots S_t^k$ admit almost-conformal embeddings into $S_\infty^1, \ldots S_\infty^k$. A sufficiently small choice of δ in $\mathcal{T}_\delta$ in *Step 3* ensures the images are "sufficiently large" (Lemma 6.13). By using results from §4, the almost-conformal maps of *Step 2* can be further adjusted to give almost-conformal maps $f_i : S_t^i \to Y_t^i$ for each $1 \leq i \leq k$. Finally, by the quasiconformal extension Lemma 3.6, these can be adjusted along the boundaries such that they fit continuously to produce an almost-conformal map $f : X_t \to Y_t$ between the glued-up subsurfaces.

6. Proof of Theorem 1.1

We shall follow the outline in the previous section, and refer to that for notation and the setup.

We begin by associating a non-negative real number (a "residue") to each topological end of the infinitely-grafted surface X_∞.

6.1. **End data.** Let $S^1, S^2, \ldots S^k$ be the components of the metric completion $\widehat{X \setminus \lambda}$.

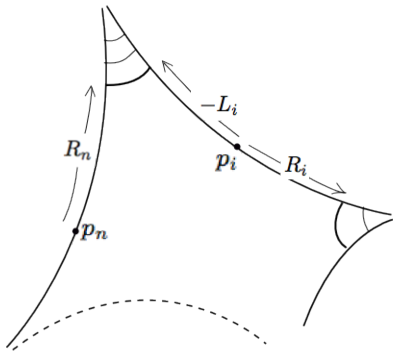
Recall that for any $1 \leq j \leq k$, a component of the boundary of S^j is either *closed* (a geodesic circle) or is *polygonal* (a geodesic circle with "spikes" - this is also called a "crown" in [CB88]).

Consider a polygonal boundary consisting of a cyclically ordered collection of n bi-infinite geodesics $\{\gamma_1, \gamma_2, \ldots \gamma_n\}$.

Choose basepoints $p_i \in \gamma_i$ for each $1 \leq i \leq n$. This choice gives the following notion of "height" on each half-plane adjacent to this polygonal boundary:

Definition 6.1 (Heights)**.** For any point on the half-plane, follow the horizontal line until it hits one of the γ_i-s. The hyperbolic distance of this point from p_i (measured with sign) is the *height* of p.

Definition 6.2 (Polygonal boundary residue)**.** For each polygonal boundary as above, we associate a non-negative real number c as follows:
(1) If n is odd, $c = 0$.
(2) If n is even, choose horocyclic leaves sufficiently far out in the cusped regions between the γ_i-s. Suppose they connect a point on γ_i at height R_i with a point on γ_{i+1} at height L_{i+1}. If one cuts along these arcs to truncate the spikes, we get a boundary consisting of alternating geodesic and horocyclic sides, where the geodesic side lying on γ_i has length $R_i - L_i$. Then define $c = |\sum\limits_{i=1}^{n}(-1)^{i+1}(R_i - L_i)|$.
We call this the *residue* for the polygonal boundary, in analogy with the "metric residue" of a planar end (see Definition 2.1).



Remark. It is easy to see that the definition in (2) above is independent of the choice of base-points: a different choice of p_i will increase (or decrease) both L_i and R_i by the same amount, and the difference remains the same. Moroever, the definition is independent of the choice of horocyclic leaves of the truncation, as the next lemma shows.

Lemma 6.3. *Let the residue for a polygonal boundary be C. Then for any H sufficiently large, there is a choice of basepoints, and a choice of horocyclic arcs such that the set of lengths of the n geodesic sides after truncation is $\{H, H, \ldots H, H + C\}$.*

Proof. Fix a basepoint p_1 on γ_1. For any H sufficiently large, there is a horocyclic leaf at distance H from p_1 on γ_1. Following that leaf, we get to γ_2, and we pick a basepoint $p_2 \in \gamma_2$ such that the horocylic leaf was at $-H/2$ on γ_2. Here H is chosen sufficiently large so that there is a horocylic leaf at $H/2$ on γ_2, and we keep following horocylic leaves and picking basepoints on each successive γ_i which form the midpoints of segments of length H, until we come back to a point p^* on γ_1 (the end point of the horocylic leaf at $H/2$ on γ_n).
Let C be the distance between p_1 and p^* on γ_1. When n is odd, this distance can be reduced to 0 by moving our initial choice of basepoint p_1 (p^* moves in a direction opposite to p_1). This uses the following observation: if γ, γ' are geodesics in the hyperbolic plane with a common endpoint ξ at the boundary at infinity, and h, h' are two horocycles also based at ξ, then the distance between $\gamma \cap h$ and $\gamma \cap h'$ is the same as the distance between $\gamma' \cap h$ and $\gamma' \cap h'$. This shows that not only do the points move in the same or different direction when the basepoint is moved, but they do so isometrically.
When n is even, this distance is independent (and equal to C) for any initial choice of basepoint p_1. In particular, if we chose the basepoint on γ_n to be the midpoint of a segment of length $(H + C)/2$ instead of $H/2$, the point p^* coincides with p_1. □

Now consider the conformal limit as in (7). For each $1 \leq j \leq k$ the surface S^j_∞ has cylindrical ends, corresponding to the closed boundary components of S^j, or flaring "planar" ends corresponding to the polygonal boundaries, or both.

Definition 6.4 (End data)**.** For each end, we can associate a *residue* which for a closed boundary equals its length, and is a non-negative number as in Definition 6.2 for a polygonal boundary.

Definition 6.5 (H-truncation)**.** Let C be the residue of an end of S^j_∞, for some $1 \leq j \leq k$, as defined above. A *truncation of the end* at height H will be the surface obtained as follows: for a polygonal boundary of S^j choose a truncation of the spikes such that the geodesic sides have lengths $\{H, H, \ldots H + C\}$ where C is the residue for the end (see Lemma 6.3), and append Euclidean rectangles of horizontal widths $H/2$ along each. For each closed boundary component, we append a Euclidean annulus of width $H/2$.

Note that the complement of the truncation of an end is a punctured disk.

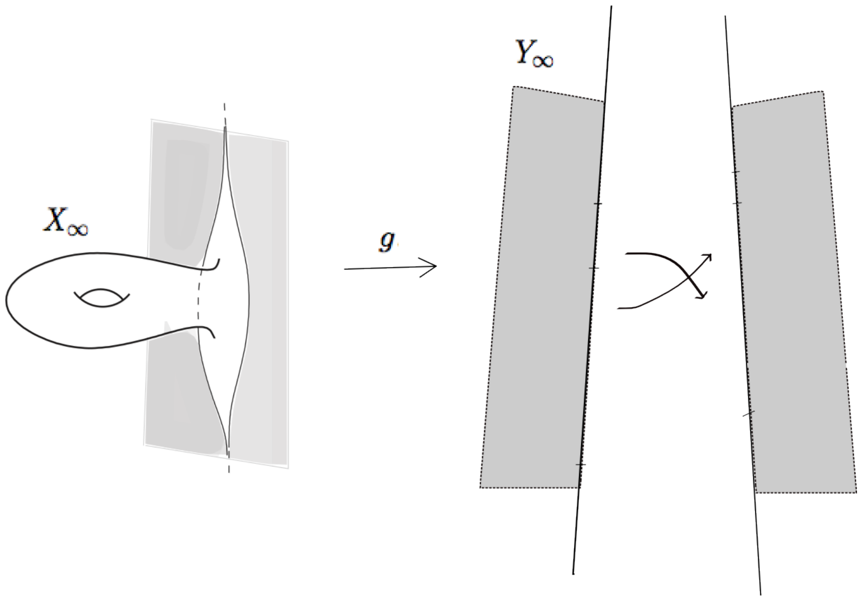


FIGURE 9. In *Step 1* one finds a conformal homeomorphism g from the infinitely grafted surface to a generalized half-plane surface Y_∞ with identical "residues".

6.2. **Step 1: The surface Y_∞.** The task at hand is to define an appropriate singular flat surface Y_∞ that is conformally equivalent to X_∞, that shall be the conformal limit of the asymptotic Teichmüller ray.

For each $1 \leq j \leq k$, consider the surface S^j_∞ (see (7)). This is a Riemann surface with punctures $p_1, \ldots p_m$ (corresponding to the planar or cylindrical ends). As in the previous section, these have end data $c_1, \ldots c_m$. We shall now equip S^j_∞ with a meromorphic quadratic differential that induces a conformally equivalent singular flat metric with a "(generalized) half-plane structure" (see Definition A.1 in the Appendix):

Namely, by Theorem A.7 there exists a singular flat surface Y^j_∞ that is conformally equivalent to S^j_∞ by a map g^j and has planar ends corresponding to the punctures with metric residues $c_1, \ldots c_m$, such that the conformal homeomorphism:

$$g^j : S^j_\infty \to Y^j_\infty$$

preserves the punctures.

We define

$$Y_\infty = Y^1_\infty \sqcup Y^2_\infty \sqcup \cdots \sqcup Y^k_\infty \tag{8}$$

and the union of the maps g^j above gives a conformal equivalence:

$$g : X_\infty \to Y_\infty \tag{9}$$

Remark. The Gauss-Bonnet theorem rules out the possibility of the exceptional cases of Theorem A.7: namely, there cannot be a hyperbolic surface with two infinite geodesics bounding a "bigon" , nor can their be two distinct closed geodesics bounding an annulus.

6.3. **Step 2: Adjusting the map g.** Consider the surfaces S^j_∞ and Y^j_∞ as in the previous section, for some $1 \leq j \leq k$.

Note that the topological ends of Y^j_∞ are "planar ends" as in Definition 2.1 or cylindrical ends (in case λ has a simple closed curve as a component). The adjustment of g in this section shall give controlled almost-conformal maps between "truncations" of S^j_∞ and Y^j_∞, and we define this adjustment according to the type of end.

Planar ends. We shall assume for convenience of notation that there is exactly one planar end. In the case of several such ends the following arguments are applied to each independently.

Let S^j_H be the H-truncation of S^j_∞ (see Definition 6.5) and Q^j_H denote its complement, that is, $Q^j_H = S^j_\infty \setminus S^j_H$.

Let P^j_H be the planar end corresponding to Q^j_H obtained by collapsing the hyperbolic part of S^j_H. Since by the construction in the previous section, the metric residues of S^j_∞ and Y^j_∞ are the same, for large enough H, P^j_H embeds isometrically in Y^j_∞.

We start by the observation that for large enough H, the surface Q^j_H admits a controlled map to this planar end:

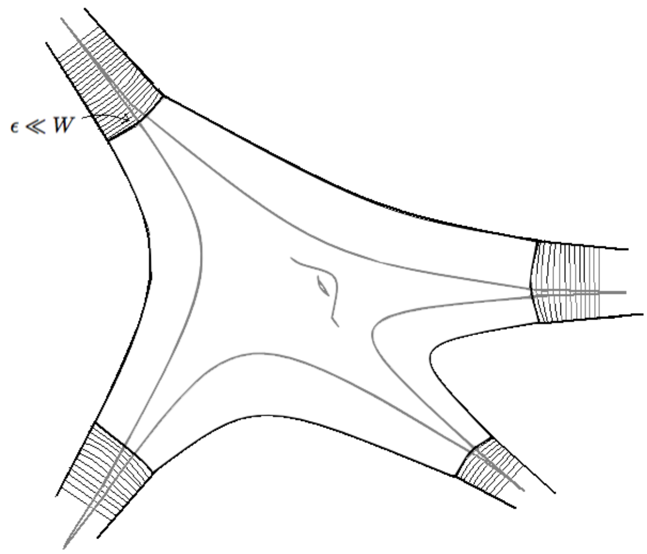


FIGURE 10. This shows Q^j_t with its polygonal boundary, and the infinite strips of width W adjoining each horizontal edge. In Lemma 6.6 the map f^j collapses the ϵ-thin hyperbolic "spikes" and is an isometry on the half-planes in the complement.

Lemma 6.6. *Fix an $\epsilon > 0$ (as in §5). Then there exists an $H_0 > 0$ such that there is a $(1+\epsilon)$-quasiconformal map*

$$f^j : Q^j_{H_0} \to P^j_{H_0}$$

that is height-preserving, and (ϵ, ϵ)-almost-isometric on any horizontal segment in the domain.

Proof. Suppose S^j has a polygonal end with n geodesic lines as boundary components, so S^j_∞ has n half-planes attached along them. The subset $Q^j_{H_0}$ has n Euclidean regions, each isometric to a "notched" half-plane (see Definition 2.1). These regions are arranged in a cycle with adjacent ones glued to each other by a thin hyperbolic "spike" of width less than ϵ (for sufficiently large H_0). This Euclidean-and-hyperbolic metric on $Q^j_{H_0}$ is C^1.

The map h^j is obtained essentially by collapsing these spikes. Divide $Q^j_{H_0}$ and $P^j_{H_0}$ into vertical strips and half-planes as shown in Figure 14. Every vertical edge of $\partial Q^j_{H_0}$ is part of a vertical line bounding a half-plane which is mapped isometrically to the corresponding half-plane in $P^j_{H_0}$. The remaining infinite strips adjoining each horizontal edge is mapped by a height preserving affine map to the corresponding strips in $P^j_{H_0}$.

Assume that H_0 is sufficiently large such that each of these horizontal edges have Euclidean length $W = 2H_0 \gg \epsilon$. The horizontal stretch factor for the affine map of the strips is then close to 1, and since it is height-preserving it preserves vertical distances. Being height-preserving the affine map agrees with the map on the half-planes previously described, and thus we have a C^1 map h^j that is close to being an isometry, and is hence almost-conformal. It is easy to check that a stretch map between two sufficiently long intervals whose lengths differ by ϵ, is an (ϵ, ϵ)-almost-isometry in the sense of Definition 3.4. □

Next we adjust the conformal map $g : S^j_\infty \to Y^j_\infty$ to an almost-conformal map that agrees with the map f^j obtained from the Lemma above, in the complement of a large enough truncation:

Proposition 6.7. *There exists $H_1 \gg H_0$, such that there is a $(1+C\epsilon)$-quasiconformal map*

$$h^j : S^j_\infty \to Y^j_\infty \tag{10}$$

that agrees with f^j on $Q^j_{H_1}$ and g^j on $S^j_{H_0}$.
(Here $C > 0$ is a universal constant.)

Proof. We shall use the quasiconformal interpolation lemma of §4.

Uniformize $S^j_{H_0} \cup \{\infty\}$ to a unit disk $\mathbb{D}$ by a conformal map ϕ that takes ∞ to 0, and $P^j_{H_0} \cup \{\infty\}$ to a unit disk $\mathbb{D}$ by a conformal map ψ that takes ∞ to 0.

By Lemma 6.6 there is a $(1+\epsilon)$-quasiconformal map $\psi \circ f^j \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$ that preserves 0.

The conformal homeomorphism $g^j : S^j_\infty \to Y^j_\infty$ preserves the punctures (at ∞), and hence the corresponding ends. For r sufficiently small, the conformal map $\psi \circ g^j \circ \phi^{-1}|_{B_r}$ is well-defined and maps the subdisk $B_r \subset \mathbb{D}$ into $\mathbb{D}$. Note that $\psi \circ g^j \circ \phi^{-1}(0) = 0$.

Applying Lemma 4.5 there is a sufficiently small $r' < r$ and a $(1+C\epsilon)$-quasiconformal map $F : B_r \to g(B_r)$ that interpolates between $\psi \circ g^j \circ \phi^{-1}$ on ∂B_r to the map $\psi \circ f^j \circ \phi^{-1}$ on $B_{r'}$.

Let H_1 be sufficiently large such that $\phi(Q^j_{H_1}) \subset B_{r'}$. Together with the restriction of g^j on $S^j_{H_0} \cup \phi^{-1}(\mathbb{D} \setminus B_r)$, the map $\psi^{-1} \circ F \circ \phi$ on $\phi^{-1}(B_r)$ defines a homeomorphism $h^j : S^j_\infty \to Y^j_\infty$ that is $(1+C\epsilon)$-quasiconformal, and satisfies the requirements of the lemma. □

Cylindrical ends. For the cylindrical ends of S^j_∞ and Y^j_∞, the adjustment of g is easier, and is identical to the discussion in §5 of [Gupa]. In particular, we have the following analogue of Proposition 6.7.

Let S^j_H denote the surface obtained by truncating the cylindrical end of S^j_∞ at height H, and let Q^j_H denote the half-infinite cylinder in its complement. The surface Y^j_∞ has a corresponding cylindrical end of identical circumference, since the residues match by the construction in §6.2 (Step 1).

Proposition 6.8 (Lemma 5.3 of [Gupa]). *There exists $H_0 > 0$ and $H_1 \gg H_0$, such that there is a $(1 + C\epsilon)$-quasiconformal map*

$$h^j : S^j_\infty \to Y^j_\infty \tag{11}$$

that is an isometry on $Q^j_{H_1}$ and restricts to g^j on $S^j_{H_0}$.
(Here $C > 0$ is a universal constant.)

6.4. **Step 3: Defining the Teichmüller ray.** Recall from Lemma 3.9 that there is the corresponding decomposition of the grafted surface at time t:

$$X_t = S^1_t \sqcup S^2_t \sqcup \cdots \sqcup S^k_t \tag{12}$$

where each S^j_t is obtained by appending halves of adjacent branches of $\mathcal{T}_\delta$ adjacent to the components of $X \setminus \mathcal{T}_\delta$. (The choice of δ shall be clarified in Lemma 6.13, the discussion in this section holds for any such choice.)

The goal of this subsection is to define the Teichmüller ray in the direction of λ. This is done by piecing together certain truncations of the (generalized) half-plane surfaces $Y^1_\infty, \ldots, Y^j_\infty, \ldots Y^k_\infty$ we obtained in (8), in the same pattern as the S^j_t-s in (12).

Definition 6.9. A *truncation* of a generalized half-plane surface Y (as in Definition A.1) is the singular flat surface with "polygonal" boundary obtained by truncating the infinite edges of the metric spine of Y, and gluing Euclidean rectangles along each non-closed side, and Euclidean annuli along each closed side. The complement of a truncation then is a disjoint union of planar ends and half-infinite cylinders.

For any $1 \le j \le k$ consider the surface S^j_t as in (12). This surface comprises several rectangles and annular regions glued along the boundary of the hyperbolic surface S^j (see §3.4) with cusps truncated along some horocycles. Consider a truncation of Y^j_∞ that involves rectangles and annuli of same vertical heights, and same *Euclidean* widths, glued in the same pattern as S^j_t. This is possible because the metric residues of the ends of S^j_∞ are equal to those of Y^j_∞ - in particular, this ensures:
(a) The Euclidean rectangles of vertical lengths $\{H, H, \ldots, H, H + C\}$ in an H-truncation of S^j_t as in Definition 6.5 glue up on the corresponding truncation of Y^j_∞ (see Theorem 2.6 of [Gupb]).
(b) The Euclidean cylinders on S^j_t have the same circumferences as the corresponding cylinders on Y^j_∞.
The resulting singular flat surface Y^j_t can be thought of as obtained by collapsing the hyperbolic part of S^j_t. In particular, Y^j_t is homeomorphic to S^j_t.

Definition 6.10 (Y_t)**.** The closed singular-flat surface Y_t is obtained by gluing the singular flat surfaces $Y^1_t, Y^2_t, \ldots Y^k_t$ along their boundaries according to the gluing of $S^1_t, \ldots S^j_t$ on X_t. Namely, any point of the boundary of S^j_t has a vertical coordinate determined by its height (Definition 6.1) and a horizontal coordinate equal to the Euclidean distance from the geodesic boundary of S^j (this is zero for a point on the horocyclic sides). Similarly we have horizontal and vertical coordinates on ∂Y^j_t determined by the Euclidean metric (the metric spine has zero horizontal coordinate). The gluing maps between the Y^j_t in these coordinates is then identical to those between the S^j_t on X_t. It is not hard to see that these gluing maps between Y_t^j-s are isometries in the singular flat metric, and one obtains a closed singular-flat surface Y_t. Moreover, Y_t acquires a marking from the marking on X_t.

Summarizing, we have the decomposition:

$$Y_t = Y_t^1 \sqcup Y_t^2 \sqcup \cdots \sqcup Y_t^k. \tag{13}$$

Definition 6.11 ($\mathcal{F}_t$)**.** The vertical foliation on the rectangles and annuli for each $Y_t^1, \ldots Y_t^k$ induce a measured foliation $\mathcal{F}_t$ on Y_t: namely, since the boundary maps are isometries the vertical lines on each rectangle and concentric circles (meridians) on each annulus glue up to give a singular foliation $\mathcal{F}$ with a transverse measure given by horizontal length in the singular-flat metric.

Lemma 6.12. *For any $t > 0$, $\mathcal{F}_t$ is measure-equivalent to $t\lambda$. In particular, the surfaces Y_t lie along a Teichmüller geodesic ray in the direction of λ.*

Proof. By construction, we have:
(1) the horizontal widths of the pieces in (13) are given by the transverse measure on $t\lambda$, and
(2) the gluing maps along their boundaries are isometries in the singular-flat metric.

To prove measure-equivalence, one needs to check that the horizontal length of any non-trivial simple closed curve γ on Y_t is the same as its transverse measure on X_t:
Consider a representative of γ on X_t of minimal transverse measure. It cuts across several constituent rectangles and annuli of the decomposition $S_t^1, \ldots S_t^j$, and can be homotoped to consist of an concatenation of embedded segments on their boundaries. Let its total transverse measure be m. The corresponding image of γ on Y_t is in the same homotopy class, and has the same horizontal length m by (1) and (2) above. Conversely, any representative of γ on Y_t of horizontal length h has a corresponding representative on X_t of equal transverse measure.
This proves that the vertical foliation $\mathcal{F}_t$ on Y_t is measure-equivalent to $t\lambda$.

Along the grafting ray, the Euclidean widths of the pieces S_t^j scale by a factor t, and consequently so do the horizontal widths of the corresponding pieces Y_t^j, whereas the vertical lengths remain fixed. A similar argument, using (2) above, shows that the *horizontal foliation* $\mathcal{H}$ on Y_t (induced from the horizontal lines on each rectangle and longitudinal lines on each cylinder) remains fixed for all $t > 0$.

Hence the surface Y_t is determined by the pair of transverse measured foliations $(t\lambda, \mathcal{H})$. By the definition of a Teichmüller ray (see §2.5), Y_t is the surface a distance $\frac{1}{2}\ln t$ along the Teichmüller ray from X in the direction λ. □

6.5. **Step 4: Constructing the map f.** The goal is to construct an almost-conformal map $f : X_t \to Y_t$.

Mapping the pieces. Recall the decomposition of the surfaces $X_t = S_t^1 \sqcup \cdots \sqcup S_t^k$ and $Y_t = Y_t^1 \sqcup \cdots \sqcup Y_t^k$ as in the previous section, and $X_\infty = S_\infty^1 \sqcup S_\infty^2 \sqcup \cdots \sqcup S_\infty^k$ as in (7).

These decompositions were dependent on the choice of a train-track neighborhood $\mathcal{T}_\delta$ (see §3.1) and it is in the following lemma that our choice of δ is specified. This lemma is a refinement of Lemma 3.9 proved in §3.4.

Lemma 6.13. *There is a (sufficiently small) $\delta > 0$ in the construction of $\mathcal{T}_\delta$ such that for all sufficiently large t, for each $1 \leq j \leq k$, there is a $(1 + C\epsilon)$-quasiconformal embedding*

$$e^j : S_t^j \to S_\infty^j \tag{14}$$

that is height-preserving on the vertical sides of the boundary and $(\epsilon, 3\epsilon)$-almost-isometric on the horizontal sides. Moreover, the image contains a truncation $S_{H_1}^j$ of the surface S_∞^j.

Proof. S_t^j consists of the hyperbolic surface $\widehat{S^j}$ that is a truncation of S^j along horocyclic arcs, with adjacent rectangles through which leaves of the lamination λ pass, and which thus have a grafted Euclidean part. For sufficiently small $\delta > 0$ in the construction of $\mathcal{T}_\delta$, the heights of these rectangles are greater than H_1 by Lemma 3.1.

By Lemma 3.7, for sufficiently large t each of these rectangles admit almost-conformal maps to Euclidean rectangles of identical heights and *Euclidean* widths, that are (ϵ, ϵ)-good on the boundary. Together with an isometry on $\widehat{S^j}$ these maps can be pieced together to give the required embedding e^j in S_∞^j. The image of this map comprises Euclidean rectangles and annuli appended to each geodesic side of $\widehat{S^j}$.

By Lemma 3.2 for sufficiently large t the Euclidean widths of the appended rectangles or annuli are also all greater than H_1. Hence the embedded image in S_∞^j contains the truncation $S_{H_1}^j$.

A horizontal side of S_t^j consists of two horizontal sides of adjacent rectangles, separated by a short horocyclic arc. The embedding is (ϵ, ϵ)-almost isometric on the sides of each rectangle (Lemma 3.7), and isometric on the intervening horocyclic arc, and hence the concatenation is $(\epsilon, 3\epsilon)$-almost-isometric. □

Proposition 6.14. *For sufficiently large t, and for each $1 \leq j \leq k$, there is a $(1 + C\epsilon)$-quasiconformal map*

$$F^j : S_t^j \to Y_t^j$$

that is $(\epsilon, 4\epsilon)$-almost-isometric on the boundary.

Proof. By the previous lemma, for sufficiently large t, we have an almost-conformal embedding $e^j : S_t^j \to S_\infty^j$ such that the image contains the truncation $S_{H_1}^j$. Postcomposing with h^j of Propositions 6.7 and 6.8 (restricted to this image), we get a $(1 + C\epsilon)$-quasiconformal embedding $F^j : S_t^j \to Y_\infty^j$. Now e^j is height-preserving, and so is h^j (since it restricts to f^j, see Lemma 6.6). In other words, they preserve the "horizontal" direction. Hence the image of F^j is the truncation Y_t^j of Y_∞^j. Moreover, since the additive errors of almost-isometries add up under composition, the composition yields an $(\epsilon, 4\epsilon)$-almost-isometry on the horizontal sides. □

The final map of the grafted surface.

Proposition 6.15. *For sufficiently large t, there is a $(1 + C\epsilon)$-quasiconformal homeomorphism $f : X_t \to Y_t$, homotopic to the identity map.*

Proof. By Proposition 6.14 for sufficiently large t we have an almost-conformal map

$$F^j : S_t^j \to Y_t^j$$

for each $1 \leq j \leq k$ that is $(\epsilon, 4\epsilon)$-almost-isometric on the boundary.

The surface X_t is obtained by gluing the S_t^j-s (12) and the surface Y_t is obtained by gluing the singular flat pieces Y_t^j (13). However the boundary maps on the pieces from F^j above differ from the gluing maps on the boundary. For the pieces to fit together continuously along the boundary, one has to post-compose with a "correcting map" $F_\partial^j : \partial Y_t^j \to \partial Y_t^j$. Since all the boundary-maps constructed so far are $(\epsilon, M\epsilon)$-good (for $M > 0$ a universal constant), so is F_∂^j. Recall that Y_t^j comprises a collection of rectangles and annuli - for the rectangles, by Lemma 3.6, for sufficiently large t one can extend F_∂^j to an almost-conformal self-map of Y_t^j. One can now adjust the maps F^j on the rectangles by post-composing with these almost-conformal maps. The composition then restricts to the desired map on the boundary. (For the annuli in Y_t^j the gluing maps are already isometries by Proposition 6.8 and no further adjustment is required.) These adjusted almost-conformal maps glue up to define an almost-conformal map $f : X_t \to Y_t$. Since the conformal homeomorphism g in (9) is homotopic to the identity map, and the gluings of the truncations preserve the marking, the final map is homotopic to the identity map as claimed. □

This completes the proof of Theorem 1.1 (see the discussion in §5).

By the parametrization mentioned in the proof of Lemma 6.12, if $Teich_{t\lambda}Y$ denotes the surface along the Teichmüller ray from Y in the direction determined by λ, we in fact have:

$$d_{\mathcal{T}}(gr_{e^{2t}\lambda}X, Teich_{t\lambda}Y) \to 0. \tag{15}$$

Appendix A. Generalized half-plane surfaces

In this section we briefly recall the work in [Gupb] and prove Theorem A.7 that is used in the proof of Theorem 1.1 (see §6.2).

One difference with the work in [Gupb] is that we aim to produce meromorphic quadratic differentials with only prescribed order and residue at the poles, discarding the (coordinate-dependent) notion of "leading order terms" at the poles.

Notation. As a minor change of convention from [Gupb], in this paper we have switched what we call the "horizontal" and "vertical" directions for the quadratic differential metric (see §2.2 for definitions). In particular, the Euclidean "half-planes" below should be thought of as those bounded by a *vertical* line on the plane, and the foliation by straight lines parallel to the boundary is its *vertical* foliation.

A.1. **Definitions.**

Definition A.1 (Half-plane surface)**.** Let $\{H_i\}_{1\leq i\leq N}$ be a collection of $N \geq 2$ Euclidean half planes and let $\mathcal{I}$ be a finite partition into sub-intervals of the boundaries of these half-planes. A *half-plane surface* Σ is a complete singular flat surface obtained by gluings by isometries amongst intervals from $\mathcal{I}$, such that each interval is paired with a unique other interval. (Note that being complete in the metric sense implies that the resulting surface has no boundary, but it does have punctures "at infinity".)

Remark. The boundaries of the half-planes form a *metric spine* of the resulting surface, an embedded graph with an induced metric, that the punctured surface deform-retracts to. So alternatively, a half-plane surface can be thought of as a gluing of half-planes to an infinite-length metric graph to form a complete singular flat surface.

Definition A.2. A half-plane surface as above is equipped with a meromorphic quadratic differential q called the *half-plane differential* that restricts to dz^2 in the usual coordinates on each half-plane.

Definition A.3 (Local data at poles)**.** The poles of a half-plane differential q are at the "punctures at infinity" of the half-plane surface. The *residue* at a pole p is the absolute value of the integral

$$c = \int\limits_{\gamma} \sqrt{q}$$

where γ is a simple closed curve enclosing p and contained in a chart where one can define $\pm\sqrt{q}$.

Remarks. 1. A neighborhood of the poles are isometric to planar ends as in Definition 2.1. The order equals $h+2$, where h is the number of half-planes in the end, and the residue of the half-plane differential equals the metric residue as in Definition 2.1. (See Thm 2.6 in [Gupb].)

2. It follows from definitions that the local data of q at the pole also satisfy the properties:

(∗) *the residue is zero if the order is even*
(∗∗) *each order is greater than or equal to* 4*.*

A.2. **The existence result.**

Theorem A.4 ([Gupb])**.** *Given a Riemann surface* Σ *with a set of points* P *and prescribed local data* $\mathcal{D}$ *of orders, and residues satisfying (∗) and (∗∗) , there exists a half-plane surface* Σ_D *and a conformal homeomorphism* $g : \Sigma \setminus P \to \Sigma_D$ *such that the local data of the corresponding half-plane differential is* $\mathcal{D}$*.*
(The only exception is for the Riemann sphere with one marked point with a pole of order 4*, in which case the residue must equal zero.)*

Remark. The main theorem of [Gupb] in fact also prescribes the "leading order terms" at the poles, which we do not need in our application here.

This can be thought of as an existence result of a meromorphic quadratic differential with prescribed poles of order at least 4, that have a global "half-plane structure" structure as described above.

Here, we shall include poles of order two, which corresponds to half-infinite cylinders. (See Thm 2.3 in [Gupb].)

Definition A.5. A *generalized* half-plane surface is a complete singular-flat surface of infinite area obtained by gluing half-planes and half-infinite cylinders along a metric graph that forms a spine of the resulting punctured Riemann surface.

Definition A.6 (Data at a double pole)**.** A pole of order two of a generalized half-plane surface has a local expression of the form $\frac{C^2}{z^2}dz^2$ and has as an associated positive real number C, that we call its *residue*. This is independent of the choice of local coordinates, and equals $(1/2\pi)$ times the circumference of the corresponding half-infinite cylinder.

We restate the theorem mentioned in §1 slightly more precisely:

Theorem A.7. *Let* Σ *be a Riemann surface with a set* P *of* n *marked points let* $\mathcal{D}$ *be local data satisfying (∗) and (∗∗) for orders not equal to two. Then there is a corresponding generalized half-plane surface* $\Sigma_{\mathcal{D}}$ *and a conformal homeomorphism*

$$g : \Sigma \setminus P \to \Sigma_{\mathcal{D}}$$

that is homotopic to the identity map.
(The only exceptions are for the Riemann sphere with exactly one pole of order 4*, or exactly two poles of order* 2*.)*

The proof of Theorem A.4 easily generalizes to include these half-infinite cylinders and give a proof of this result. In the next section we provide a sketch of this proof, referring to [Gupb] for details.

A.3. **Sketch of the proof of Theorem A.7.** A "generalized" half-plane surface is allowed to have half-infinite cylinders, in addition to half-planes. We shall follow the outline of the proof in [Gupb] (see §5 and §12 of that paper) with the additional discussion for the poles of order two. For all details see [Gupb].

We already have a choice of coordinate charts around the points of P on Σ, since the leading order terms of the prescribed local data $\mathcal{D}$ depend on such a choice. Pick one such chart U containing the pole p, and conformally identify the pair with $(\mathbb{D}, 0)$. Let the local data associated with this pole consist of the order n and residue a.

Quadrupling. Produce an exhaustion of $\Sigma \setminus P$ by compact subsurfaces Σ_i by excising, from each coordinate disk as above, the subdisk U_i of radius 2^{-i}.

We shall put a singular flat metric on each Σ_i that we can complete to form a (generalized) half-plane surface Σ_i':

For the boundary component ∂U_i, mark off $(n-2)$ disjoint arcs on it (no arcs for order two). Take two copies of the surface Σ_i, and "double" across these boundary arcs, that is, glue the arcs on corresponding boundary components by an anti-conformal involution. If some orders are greater than or equal to 4, the doubled surface has "slits" corresponding to the complementary arcs on each of the boundary components which are not glued. Now form a *closed* Riemann surface $\widehat{\Sigma_i}$ by doubling across these slits.

Singular flat metrics. On $\widehat{\Sigma_i}$ we have disjoint homotopy classes of simple closed curves around each of the slits glued in the second doubling step (n of them associated with the pole p). By a theorem of Jenkins and Strebel (see [Str84]) there is a holomorphic quadratic differential which in the induced singular flat metric comprises metric cylinders glued together along their boundaries. Moreover, the differential is unique once we specify the "heights" of the cylinders. Quotienting back by the involutions of the two doubling steps, one gets a singular flat metric on the surface Σ_i we started with. Each metric cylinder gives a rectangle as a quotient, for higher-order poles, and an annulus for a double-order pole. Each boundary component of the singular flat surface Σ_i is either closed (for the latter case) or "polygonal" with alternating vertical and horizontal sides, corresponding to the boundary arcs we chose and their complementary arcs. For details see §6 of [Gupb].

The choice of "heights" of the cylinders gives the lengths of the "vertical" sides in the quotient. We prescribe these such that their alternating sum equals the residue for higher order poles. For a double-order pole, we choose the height (circumference of the annulus) equal to 2π times its residue (Definition A.6).

By choosing the arcs in the initial step appropriately one can prescribe the lengths of the cylinders (see §7 of [Gupb].) Together with the prescribed "heights" this gives complete control on the dimensions of these polygonal boundaries. In particular, we choose these dimensions such that the boundary component corresponding to p is isometric to the boundary of a truncation of a planar end (of residue a) at height H_i, where

$$H_i = \left(H_0 \cdot 2^i\right)^{n/2} \tag{16}$$

(H_0 is a constant that we can vary to get different "leading order terms" as in [Gupb], which we do not pursue here.)

For each, we glue in a appropriate planar end (see Definition 2.1) or half-infinite cylinder, to get a generalized half-plane surface Σ_i'.

A geometric limit. Since $H_i \to \infty$ as $i \to \infty$ in (16), the planar end one glues in to form Σ_i' gets smaller as a conformal disk, and it is easy to check from Corollary 4.7 that this implies that for any $\epsilon > 0$, for sufficiently large i there is a $(1+\epsilon)$-quasiconformal homeomorphism $f_i : \Sigma_i \to \Sigma \setminus P$ preserving the markings.

On the other hand, one can show that the sequence of $\{\Sigma_i\}_{i\geq 1}$ has a generalized half-place surface $\Sigma_{\mathcal{D}}$ as a conformal limit. As in [Gupb], the proof of this breaks into two steps:

Firstly, one can show that the corresponding sequence of meromorphic quadratic differentials q_i converges to one with the same local data $\mathcal{D}$, in the meromorphic quadratic differential bundle over $\mathcal{T}_g$ corresponding to the associated divisor. This holds because of the geometric control in (16) - this makes the planar end $\mathcal{P}_{H_i}$ glued in conformally comparable to the subdisk of radius 2^{-i} excised from U (see Lemma 7.4 of [Gupb]). Namely, there is an almost-conformal map between the pairs $(\mathbb{D}, B_{1/2^i})$ and $(U, \mathcal{P}_{H_i})$. On the (generalized) half-plane surface Σ_i', the disk U can be identified with a planar domain via a conformal embedding f_i. Moreover the subdisk U_i is the preimage by f of the planar end $\mathcal{P}_{H_i}$. The meromorphic quadratic differential q_i is then a pullback of some fixed differential q_0 on $\mathbb{C}$ by f_i (see the remark following Definition 2.1). By the almost-conformal correspondence above, there is a control on diameters of the planar domains involved, which gives a uniform bounds of the derivative of f_i at p. The sequence f_i then forms a normal family, and $f_i \to f$ and $q_i \to q$. On U, q restricts to the pullback of q_0 by f.

In the second step, one shows that q is also a generalized half-plane differential, that is, the sequence Σ_i' in fact converges to a generalized half-plane surface. This latter limit $\Sigma_{\mathcal{D}}$ is built by attaching half-planes and half-infinite cylinders along a metric spine that the metric spines of Σ_i' converge to, after passing to a subsequence. One needs an argument to show that this limiting spine has the same topology. It suffices to show that one can choose a subsequence such that the metric graphs are identical as *marked graphs*: Any edge of a metric spine along the sequence of converging q_i has an adjacent collar of area proportional to its length, and along the sequence they cannot accumulate an increasing amount of "twists" about any non-trivial simple closed curve since that contributes an increasing q-area to an embedded annulus about the curve. A sequence of spines having bounded twists about any simple closed curve is topologically identical after passing to a subsequence. Any cycle in the metric graph will then have a lower bound on its q_i-length (since the q_i-s are converging). Hence no cycles collapse, and $\Sigma_{\mathcal{D}}$ has the same topology. Almost conformal maps can then be built to the limiting surface by "diffusing out" any collapse of a sub-graph that is tree (or forest), to the adjacent half-planes. Namely, for any $\epsilon > 0$, for all sufficiently large i we obtain a $(1+\epsilon)$-quasiconformal homeomorphism $g_i : \Sigma_i' \to \Sigma_{\mathcal{D}}$ that preserves the marking.

Standard compactness results of quasiconformal maps then implies that the sequence $g_i \circ f_i^{-1}$ converges to a homeomorphism $g : \Sigma \setminus P \to \Sigma_{\mathcal{D}}$ preserving the markings, that is $(1+\epsilon)$-quasiconformal for any $\epsilon > 0$, and is hence conformal.

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