Math 120: Practice for the final

Setting up integrals

Q1. Set up the surface integral

$$\iint_S dS$$

for the following surfaces, by treating each either as a graph or a parametric surface:

(1) S is part of the cone $z = \sqrt{x^2 + y^2}$ which lies between the planes $z = 0$ and $z = 4$.

(2) S is part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 3$.

(3) S is the planar triangular region that is bounded by the three line segments from $(0, 0, 3)$ to $(2, 0, 0)$, from $(2, 0, 0)$ to $(0, 1, 0)$ and from $(0, 1, 0)$ to $(0, 0, 3)$

(4) S is the region on the plane $x + z = 2$ that lies inside the cylinder $x^2 + y^2 = 1$.

(5) S is the region on the plane $x + z = 2$ that lies inside the cylinder $x^2 + y^2 = 2x$.

(6) S is part of the sphere $x^2+y^2+z^2 = 1$ that lies above the plane $z = 1/\sqrt{2}$.

Q2. Set up the triple integral

$$\iiint_E dV$$

for the following solid regions:

(1) E is the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = 0$, $z = 0$ and $z = 5$.

(2) E is the region bounded by the parabolic cylinder $y = 1 - x^2$, and the planes $y = 0$, $z = 0$ and $z = 4$.

(3) E is the region bounded by the coordinate planes and the plane $2x+y+z=1$.

(4) E is the region inside the surface given by the equation $x^2+y^2+z^2=2z$ and above the cone given by $\phi=\pi/3$ in spherical coordinates.

(5) E is the region bounded by the parabolic cylinder $y=x^2-1$ and the planes $x-y+1=0$, $z=0$ and $z=4$.

(6) E is the region below the cone $z=\sqrt{x^2+y^2}$ that lies inside the cylinder $x^2+y^2=1$.

(7) E is the region below the plane $x+z=2$ that lies inside the cylinder $x^2+y^2=2x$.

Solutions:

Note: I haven't included accompanying pictures - please try and sketch them on your own.

Q1.
(1) Treat S as a graph. Its "shadow" D on the xy-plane is the disk of radius 4. While converting from dS to dA, you need to calculate:

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=\sqrt{2}$$

and so the surface integral is:

$$\iint_S dS=\iint_D \sqrt{2}dA=\int_0^{2\pi}\int_0^4 \sqrt{2}rdrd\theta$$

since an integral over a disk centered at the origin is best done in polar coordinates.

(2) Treat S as a graph. Solving the equations $z=x^2+y^2$ and $x^2+y^2=3$ we easily see that the intersection of the paraboloid and the cylinder is at $z=3$, so the radius of the "shadow" D of S on the xy-plane is a disk of radius $\sqrt{3}$. We calculate:

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+(2x)^2+(2y)^2}=\sqrt{1+4x^2+4y^2}$$

and so the surface integral is:

$$\iint_S dS=\iint_D \sqrt{1+4x^2+4y^2}dA=\int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1+4r^2}rdrd\theta$$

(3) Since S is part of a plane, the first step is to find the equation of the plane and then treat S as a graph.
The plane has equation $ax + by + cz = d$ and it is easy to figure out that $3x + 6y + 2z = 6$ passes through the points $(0, 0, 3)$, $(2, 0, 0)$ and $(0, 1, 0)$.
Expressing z as a function of x and y we see that S is the graph $z = 3 - \frac{3}{2}x - 3y$.
Its "shadow" D is a triangle on the xy-plane, bounded by the x and y axes and the line segment from $(2, 0, 0)$ to $(0, 1, 0)$, which has equation $y = -x/2 + 1$.
We calculate:

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(-\frac{3}{2}\right)^2 + (-3)^2} = \frac{7}{2}$$

and so the surface integral is:

$$\iint_S dS = \iint_D \frac{7}{2} dA = \int_0^2 \int_0^{-x/2+1} \frac{7}{2} dy dx$$

(4) Treat S as the graph $z = 2 - x$. Since we are only concerned with the portion inside the cylinder, we see that its "shadow" D is the disk $x^2 + y^2 \leq 1$ on the xy-plane. We calculate:

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + (-1)^2 + (0)^2} = \sqrt{2}$$

and so the surface integral is:

$$\iint_S dS = \iint_D \sqrt{2} dA = \int_0^{2\pi} \int_0^1 \sqrt{2} r dr d\theta$$

(5) The equation $x^2 + y^2 = 2x$ can be rewritten as $(x - 1)^2 + y^2 = 1$ and is hence the cylinder of radius 1 around the line parallel to the z-axis through $(1, 0)$. One can treat the surface as the graph of $z = 2 - x$ over this "shadow" D (see the previous problem also!).
The double integral is

$$\iint_S dS = \int_0^{\pi} \int_0^{2\cos\theta} \sqrt{2} r dr d\theta$$

(6) S is part of a sphere, so it is best treated as a parametric surface $\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$ where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/4$. We get the range of ϕ by drawing the cross section of the picture, and some trigonometry, or by using $z = 1/\sqrt{2}$ is $\rho\cos\phi = 1/\sqrt{2}$ in spherical coordinates and $x^2+y^2+z^2 = 1$ is $\rho = 1$.
The double integral is

$$\iint_S dS = \int_0^{2\pi} \int_0^{\pi/4} \sin\phi d\phi d\theta$$

where recall that the surface area element on a sphere of radius a is $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$.

Q2.

(1) This is best done in cylindrical coordinates. The shadow of E on the xy-plane (where we shall use polar coordinates) is a half disk.
The triple integral becomes

$$\iiint_E dV = \int_0^\pi \int_0^1 \int_0^5 r dz dr d\theta$$

(2) Treat E as a "Type-1" region (between the planes $z = 0$ and $z = 5$), so you again need to figure out what the shadow D is on the xy-plane. The triple integral becomes

$$\iiint_E dV = \iint_D \int_0^4 dz dA = \int_{-1}^1 \int_0^{1-x^2} \int_0^4 dz dy dx$$

(3) Treat E as a "Type-1" region between the planes $z = 0$ and $z = 1-2x-y$, with shadow a triangle on the xy-plane. The triple integral becomes

$$\iiint_E dV = \int_0^{1/2} \int_0^{1-2x} \int_0^{1-2x-y} dz dy dx$$

(4) Notice that the surface is the sphere $x^2 + y^2 + (z-1)^2 = 1$ (take the $2z$ across and complete the square). In spherical coordinates, the surface becomes $\rho = 2\cos\phi$ (remember $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$).
The triple integral is best set up in spherical coordinates:

$$\iiint_E dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$

(5) Treat E as a "Type-1" region (between the planes $z = 0$ and $z = 4$), so you need to figure out what the shadow D is on the xy-plane.
The region D is bounded between the parabola $y = x^2 - 1$ and the line $y = x+1$ so we see that they intersect at $x = -1$ and $x = 2$. The triple integral becomes

$$\iiint_E dV = \iint_D \int_0^4 dz dA = \int_{-1}^2 \int_{x^2-1}^{x+1} \int_0^4 dz dy dx$$

(6) This is best done in cylindrical coordinates, the answer is:

$$\iiint_E dV = \int_0^{2\pi} \int_0^1 \int_0^r r dz dr d\theta$$

since remember in cylindrical coordinates the cone $z = \sqrt{x^2 + y^2}$ is $z = r$.

(7) See Q1 (5) also. The triple integral is set up in cylindrical coordinates. Note that the shadow D is a disk of radius 1 with center $(1, 0)$, which is $r \leq 2\cos\theta$ in polar coordinates.

$$\iiint_E dV = \int_0^\pi \int_0^{2\cos\theta} \int_0^{2-r\cos\theta} r dz dr d\theta$$