# Math 120: Examples

## Green's theorem

**Example 1**. *Consider the integral*

$$\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

*Evaluate it when*
*(a) $C$ is the circle $x^2+y^2=1$.*
*(b) $C$ is the ellipse $x^2+\frac{y^2}{4}=1$.*

**Solution.** (a) We did this in class. Note that

$$P = \frac{-y}{x^2+y^2}, Q = \frac{x}{x^2+y^2}$$

and so $P$ and $Q$ are *not* differentiable at $(0,0)$, so not differentiable everywhere inside the region enclosed by $C$. So we *can't* apply Green's theorem directly to the $C$ and the disk enclosed by it. (whenever you apply Green's theorem, remember to check that $P$ and $Q$ are differentiable everywhere inside the region!).

But *away* from $(0,0)$, $P$ and $Q$ *are* differentiable, and one can check that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \tag{1}$$

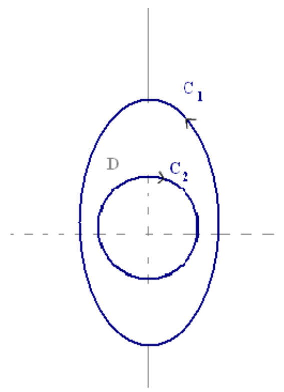
However, because of the "$x^2+y^2$" terms floating around in the integrand, it is not too hard to compute the line integral over the circle directly:
Parametrize $C$ as $(\cos t, \sin t)$ where $0 \le t \le 2\pi$.
Then

$$\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} -\sin t(-\sin t dt) + \cos t(\cos t dt) = \int_0^{2\pi} 1 \cdot dt = 2\pi.$$

(b) Notice that the ellipse $x^2+\frac{y^2}{4}=1$ lies *outside* the circle of radius 1 which we looked at in part (a). Inbetween, the two curves bound some region $D$. Notice that on $D$, our $P$ and $Q$ *are* differentiable (the only "problem point" was the origin which $D$ avoids), so we *can* apply Green's theorem to this region $D$ (with a hole) and its boundary $C'$:

$$\int_{C'} Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA = \iint_D (0) dA = 0 \tag{2}$$

where the integrand became 0 because on $D$, equation (1) holds.




Now $C'$ is composed of *two* curves: $C_1$ (our ellipse on the outside), going counter-clockwise, and $C_2$ (the circle in the inside), going *clockwise*. So

$$\int_{C'} Pdx + Qdy = \int_{C_1} \Big(Pdx + Qdy\Big) + \int_{C_2} \Big(Pdx + Qdy\Big)$$

But the integral over $C_2$ is just the integral we calculated in part (a), but with a negative sign since $C_2$ goes *clockwise* instead of *counter-clockwise*. Sot:

$$\int_{C'} Pdx + Qdy = \int_{C_1} Pdx + Qdy - 2\pi$$

But we just saw from equation (2) that

$$\int_{C'} Pdx + Qdy = 0$$

So we have

$$\int_{C_1} Pdx + Qdy = 2\pi.$$

which is exactly the integral over the ellipse that we wanted.

(*Note.* This argument actually shows that in this example the line integral over *any* closed curve about the origin would be $2\pi$!)

**Example 2**. *Use Green's theorem to evaluate the line integral*

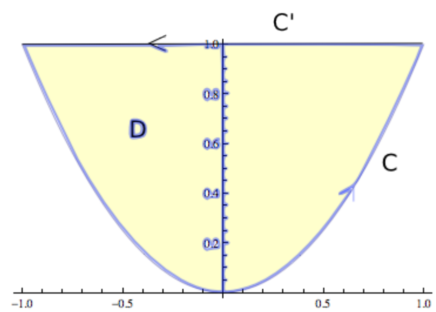
$$\int_C (1 + xy^2)dx - x^2ydy$$

*where $C$ consists of the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.*

(The terms in the integrand differs slightly from the one I wrote down in class.)

**Solution.** *Note: This line integral is simple enough to be done directly, by first parametrizing $C$ as $\langle t, t^2 \rangle$ where $-1 \leq t \leq 1$. However, we'll use Green's theorem here to illustrate the method of doing such problems.*

$C$ is not closed. To use Green's theorem, we need a closed curve, so we *close up* the curve $C$ by following $C$ with the horizontal line segment $C'$ from $(1, 1)$ to $(-1, 1)$.




The closed curve $C \cup C'$ now bounds a region $D$ (shaded yellow).

We have:

$$P = 1 + xy^2, Q = -x^2y$$

and we can calculate the partial derivatives:

$$\frac{\partial Q}{\partial x} = -2xy, \frac{\partial P}{\partial y} = 2xy$$

Applying Green's theorem to this region $D$, we get:

$$\int_{C \cup C'} (1+xy^2)dx - x^2ydy = \iint_D (-2xy - 2xy)dA = \iint_D -4xydA = \int_{-1}^{1}\int_{x^2}^{1} -4xydydx = 0.$$

(The last step involves the actual calculation using iterated integrals.)

Now parametrizing $C'$ as $\langle t, 1\rangle$ where $t$ goes from 1 to $-1$, we have:

$$\int_{C'} (1+xy^2)dx - x^2ydy = \int_1^{-1} (1+t\cdot 1^2)dt = \left(t+\frac{t^2}{2}\right)\Big|_1^{-1} = -2$$

and so:

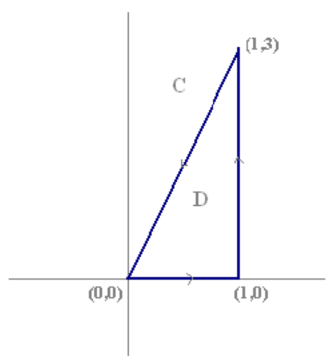
$$\int_C xy^2dx - x^2ydy = \int_{C\cup C'} - \int_{C'} = 0-(-2) = 2.$$

**Example 2**. *Use Green's theorem to evaluate*

$$\int_C \sqrt{1+x^3}dx + 2xydy$$

*where $C$ is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$ oriented clockwise.*

**Solution.** We first (as always!) draw a figure.




The curve $C$ can thought of the union of the three line segments, which can be parametrized easily, but doing the line integral directly would be hard/impossible (nasty terms like $\sqrt{1+t^3}dt$ cannot be integrated).
But, Green's theorem converts the line integral to a *double* integral over the region $D$ enclosed by the triangle, which is easier:
Let

$$P = \sqrt{1+x^3}, Q = 2xy$$

We can calculate:

$$\frac{\partial Q}{\partial x} = 2y, \frac{\partial P}{\partial y} = 0$$

Then using Green's theorem our line integral becomes

$$\int_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA = \iint_D 2y dA$$

We now need to figure out how to parametrize the region $D$ so that we can set up the limits in the double integral (it's a nice right-angled triangle, so it can't be that hard):

Think of traversing the whole region $D$ this way: let $x$ vary from 0 to 1. For each $x$, we let $y$ vary from 0 to wherever it hits the slanting side (roof!) of the triangle. Now that side is given by the equation $y = 3x$ (as you can calculate since it passes through $(0,0)$ and $(1,3)$), so the maximum $y$-height you need to go when you are at $x$, is $3x$. So with the limits, the integral becomes:

$$\int_0^1 \int_0^{3x} 2y dy dx$$

Notice that $dx$ is on the *outside* since we are varying $x$ "first" (and for each $x$, we *then* vary $y$). We can evaluate this integral easily:

$$\int_0^1 \int_0^{3x} 2y dy dx = \int_0^1 y^2 \Big|_0^{3x} dx = \int_0^1 (9x^2 - 0) dx = (3x^3)\Big|_0^1 = 3.$$

So that's the answer:

$$\int_C \sqrt{1+x^3} dx + 2xy dy = 3.$$

**Example 3**. *Use Green's theorem to find:*

$$\int_C x^2 y dx - xy^2 dy$$

*where $C$ is the circle $x^2 + y^2 = 4$ going counter-clockwise.*

**Solution.** I'll skip drawing the curve $C$: we can imagine it in our minds (just a circle of radius 2 centered at the origin, going counter clockwise). It encloses a *disk* of radius 2, which we call $D$.

To use Green's theorem, let's figure out what our $P$ and $Q$ are, and compute it's partial derivatives:

$$P = x^2 y, Q = -xy^2$$

We can calculate:

$$\frac{\partial Q}{\partial x} = -y^2, \frac{\partial P}{\partial y} = x^2$$

Then using Green's theorem our line integral becomes

$$\int_C Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA = \iint_D -y^2 - x^2 dA = -\iint_D (x^2 + y^2) dA$$

Now $D$ being a disk, and the integrand being the way it is, the integral is begging to be done in polar coordinates:

$$-\iint_D (x^2 + y^2) dA = -\int_0^{2\pi} \int_0^2 r^2 \cdot r dr d\theta$$

Notice that the upper limit for the inner integral is 2 because $r$ varies till 2 ($D$ is a disk of radius 2). Also don't forget that the area element $dA$ in polar coordinates is $r dr d\theta$!

Let's finish the calculation:

$$-\int_0^{2\pi} \int_0^2 r^2 \cdot r dr d\theta = -\int_0^{2\pi} r^4/4 \Big|_0^2 d\theta = -\int_0^{2\pi} 4 d\theta = -8\pi.$$

That's the answer:

$$\int_C x^2 y dx - x y^2 dy = -8\pi.$$

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