MATH 120: Strategies

(A Last-Minute Summary)

Lines and planes:
The equation of a **plane** passing through a point (x_0, y_0, z_0) and having normal vector $\vec{n} = \langle a, b, c\rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

and the equation of a **line** passing through a point (x_0, y_0, z_0) and having direction vector $\vec{v} = \langle a, b, c\rangle$ is given in parametric form by:

$$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc$$

Limits and continuity:
A function $f(x, y)$ is NOT continuous at a point P if *the values of the function do not agree along different paths approaching* P. In that case, the limit of the function as you approach P does NOT exist.

So if asked to find if the limit exists, one should first try the axes, lines $y = mx$ and so on. If the limits agree for all these paths, convert to polar and try to prove that the limit exists.

Partial derivatives:
Given an expression $F(x, y, z) = c$ we can think of z as *implicitly* defined as a function of x and y. To compute the partial derivatives $\partial z/\partial x$, for example, remember to treat y as a constant, and z as a function, and use the *product rule* and *chain rule*.

Product rule example: $\frac{\partial}{\partial x}(xyz) = yz + xy\frac{\partial z}{\partial x}$.
Chain rule example: $\frac{\partial}{\partial x}\sin(xy) = \cos(xy)\frac{\partial(xy)}{\partial x} = y\cos(xy)$.

Tangent planes: For a level surface $F(x, y, z) = c$ the normal vector at any point is in the direction of the *gradient vector*:

$$\vec{n} = \nabla F = \langle F_x, F_y, F_z\rangle$$

The tangent plane at a point is determined by the normal vector (and the point).

Example: The normal direction at $(1, 1, 1)$ to the surface given by $x^2 + 2y^2 + 3z^2 = 6$ is $\vec{n} = \langle 2, 4, 6\rangle$. The tangent plane has equation $2(x - 1) + 4(y - 1) + 6(z - 1) = 0$ or $x + 2y + 3z = 6$.

Critical points, second derivatives test:
For a function $f(x, y)$, the critical points are wherever both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Each critical point is either a *local maximum* or *local minimum* or a *saddle point*. One can check by computing $D = AC - B^2$, where $A = f_{xx}$, $B = f_{xy}$, and $C = f_{yy}$ and using the "D"-test:

$$D < 0 \implies \text{saddle point},\ D > 0 \implies \text{local max (when } A < 0\text{) or local min (when } A > 0\text{)}.$$

When $D = 0$, one has to figure out the nature of the critical point by examining the shape of the surface.

Absolute max and min:
Step 1. Find the critical points of $f(x, y)$ inside the region D (could be the whole plane).
Step 2. Parametrize the boundary of D (several pieces if necessary) as $\langle x(t), y(t)\rangle$ and look for the max and min of the *single-variable* function $g(t) = f(x(t), y(y))$ (here $a \le t \le b$ for some a and b).

Step 3. Compare the max and min along the boundary (step 2), and the values at the critical points (step 1) to find the absolute max and min.

Grad, curl, div:
Given a function $f(x, y)$, its *gradient vector field* is $\nabla f = \langle f_x, f_y, f_z \rangle$.

If $\nabla = \langle \partial_x, \partial_y, \partial_y \rangle$, then $curl\vec{F} = \nabla \times \vec{F}$ and $div\vec{F} = \nabla \cdot \vec{F}$, where $\vec{F}(x, y, z) = \langle P, Q, R \rangle$ is any vector field.

Fact: $F = \nabla f$ for some $f \Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for *any* closed curve $\Leftrightarrow curl\vec{F} = 0$ (and its defined everywhere). (Such a vector field $\vec{F}$ is called "conservative")

Surface integrals: Remember to think of S as either a graph over a shadow D or a parametric surface from a region D on the parameter plane. In the former case, $\vec{n} = \langle -\partial z/\partial x, \partial z/\partial y, 1 \rangle$ and in the latter $\vec{n} = \vec{r}_u \times \vec{r}_v$. (Choose $\vec{n}$ or $-\vec{n}$ depending on orientation.)
The *surface area* of S is just $\displaystyle\iint_S dS$ - for a parametric surface given by $\vec{r}(u, v)$ with a parameter domain D this is: $\displaystyle\iint_D |\vec{r}_u \times \vec{r}_v| dA$.

Triple integrals: To set up limits, decide on whether to use xyz- or spherical or cylindrical coordinates.
In *spherical* we have $\rho^2 = x^2 + y^2 + z^2$, $z = \rho \cos\phi$, $dV = \rho^2 \sin\phi d\rho d\phi d\theta$ etc.
In *cylindrical* we have $r^2 = x^2 + y^2$, $dV = r dz dr d\theta$, etc.
If E is a solid region having a "shadow" D on the xy-plane and between the graphs $z = f_1(x, y)$ and $z = f_2(x, y)$ then:

$$\iiint_E \dots dV = \iint_D \Big(\int_{f_1(x,y)}^{f_2(x,y)} \dots dz \Big) dA$$

and we use cylindrical coordinates (converting x, y to r, θ) when D is a polar region.

Big theorems:

(FTLI) $\displaystyle\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$ for a curve C from P_0 to P_1.

(Green's) $\displaystyle\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$ *(C is closed and +ly oriented)*

(Stokes) $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot d\vec{S}$ *(C is the +ly oriented boundary of S)*

(Div Thm) $\displaystyle\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV$ *(S is closed and oriented outward)*

When you see...
$\displaystyle\int_C \vec{F} \cdot d\vec{r}$ ... **think of** (a) checking if $\vec{F}$ is conservative, and using FTLI, (b) doing it directly, or (c) choosing a surface and using Stokes if C is closed , or (d) closing it up and using Stokes, subtracting the line integral on the added bit.

$\displaystyle\iint_S \vec{F} \cdot d\vec{S}$ ... **think of** (a) doing it directly, (b) using Div Thm if S is closed, or (c) closing it up and using Div Thm, subtracting the flux over the added bit.

$\displaystyle\iint_S curl\vec{F} \cdot d\vec{S}$ ... **think of** (a) using Stokes to convert it to a line integral, (b) using Stokes to replace S with a nicer surface S', or (c) doing it directly.

General advice:
1. Do each step carefully. 2. Check calculations, and 3. ***Don't panic!***