Miscellaneous examples

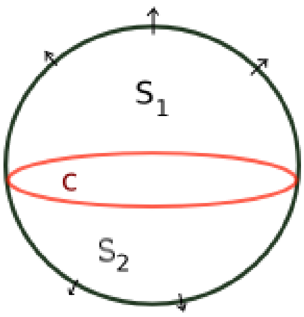
Math 120 Section 4

Stokes' theorem

Example 1. *Let $\vec{F}$ be any differentiable vector field defined in $\mathbb{R}^3$, and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented outward. Show that*

$$\iint_S curl\vec{F} \cdot \vec{n} dS = 0.$$

Solution. Divide up the sphere S into the upper hemisphere S_1 and the lower hemisphere S_2, by the unit circle C that is the "equator". Note that each hemisphere has a boundary curve (the equator C).



Applying Stokes' theorem to the surface S_1 gives:

$$\iint_{S_1} curl\vec{F} \cdot \vec{n} dS_1 = \int_{C_{ccw}} \vec{F} \cdot d\vec{r} \tag{1}$$

where C_{ccw} indicates that the curve C is oriented "counterclockwise" as seen from above. (Check that this is the positive orientation of the curve, since the normal vectors to S_1 point outward.)

Stokes' theorem for the surface S_1 gives:

$$\iint_{S_2} curl\vec{F} \cdot \vec{n} dS_2 = \int_{C_{cw}} \vec{F} \cdot d\vec{r} \tag{2}$$

where this time the boundary C is oriented *clockwise* - that is the positive orientation when the normal vectors for S_2 point "outward" (check with the Left Hand Rule!).

But we know that reversing orientation changes the sign of the line integral:

$$\int\limits_{C_{cw}} \vec{F} \cdot d\vec{r} = - \int\limits_{C_{ccw}} \vec{F} \cdot d\vec{r}$$

So on adding (1) and (2) we get:

$$\iint\limits_{S} curl\vec{F} \cdot \vec{n} dS = \iint\limits_{S_1} curl\vec{F} \cdot \vec{n} dS_1 + \iint\limits_{S_2} curl\vec{F} \cdot \vec{n} dS_2 = \int\limits_{C_{ccw}} \vec{F} \cdot d\vec{r} + \int\limits_{C_{cw}} \vec{F} \cdot d\vec{r} = 0.$$

□

Note. This argument in fact works for *any* closed surface, by dividing the surface into two using any closed curve C. In fact, we shall see another solution to this in class, using the *Divergence Theorem.*

Example 2 (Exercise 5 in Section 16.8)**.** *The surface S consists of the top and four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward. Evaluate*

$$\iint\limits_{S} curl\vec{F} \cdot \vec{n} dS$$

where $\vec{F}(x, y, z) = \langle xyz, xy, x^2yz \rangle$.

Solution. Here the curl of the vector field is:

$$curl\vec{F} = \langle x^2z, xy - 2xyz, y - xz \rangle$$

Here it is probably tedious to compute the flux of this vector field over S directly, because one has to do it separately on each of the five faces of the cube. But whenever the question asks to compute the flux of the *curl* of a vector field, one should think of Stokes' theorem, and that helps here.

Stokes theorem gives you two ways to avoid computing the flux of the curl over S:

One is to compute a line integral instead:

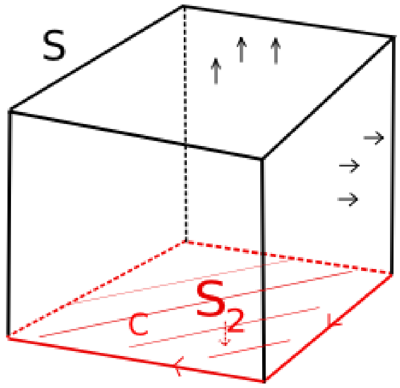
$$\iint\limits_{S} curl\vec{F} \cdot \vec{n} dS = \int\limits_{C} \vec{F} \cdot d\vec{r}$$

Here, C is the boundary of S which is a "square" on the $z = -1$ plane (check what the orientation should be, given that S is oriented outward!).

The other way (which we use here) is to *replace* S with a simpler surface S_2 that has the same boundary curve:

$$\iint\limits_{S} curl\vec{F} \cdot \vec{n} dS = \iint\limits_{S_2} curl\vec{F} \cdot \vec{n} dS_2$$

Here, we can take S_2 to be the "missing face" of the cube, which is the part of the plane $z = -1$ enclosed by the square C, with the normal vector pointing



"downward".

Computing the flux of the curl over S_2 is easy! Treat S_2 as the graph of the function $z = -1$ over the domain D which is the square $-1 \le x \le 1, -1 \le y \le 1$ on the xy-plane. The normal vectors are $-\vec{k}$ everywhere (S_2 is flat!).

So we have:

$$\iint_{S_2} curl\vec{F} \cdot \vec{n} dS_2 = \iint_D \langle x^2(-1), xy - 2xy(-1), y - x(-1)\rangle \cdot \langle 0, 0, -1\rangle dA$$

where we have plugged in $z = -1$ in the expression for $curl\vec{F}$ since we are looking along the graph of $z = -1$.

This now becomes:

$$\iint_D (-x-y)dA = \int_{-1}^{1}\int_{-1}^{1} (-x-y)dxdy = 0$$

and that's the answer!

□

Triple integrals

Example 3 (From the Final Exam, Spring '11)**.** *Convert the triple integral*

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)dzdydx$$

into cylindrical coordinates. Do not evaluate the integral.

Solution. Looking at the limits for the variables x and y and z, we first figure out that the solid region is the one between the graphs $z = \frac{1}{2}$ and $z = \sqrt{1-x^2-y^2}$ over a disk D of radius $\frac{\sqrt{3}}{2}$ on the xy-plane centered at $(0, 0)$. (See the figure!)

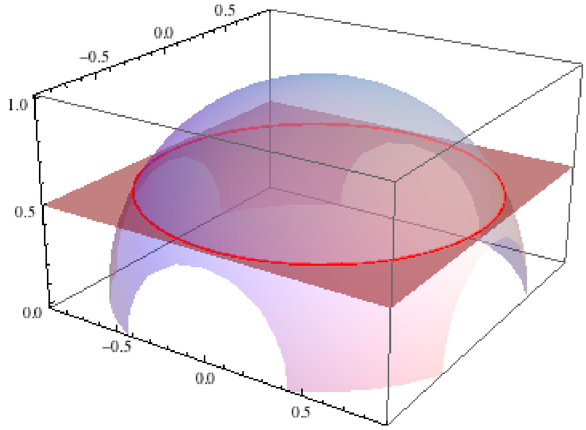


Figure 1: The plane $z = \frac{1}{2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle of radius $\frac{\sqrt{3}}{2}$. Our solid region is the one between the plane and the sphere, and its "shadow" is a disk of radius $\frac{\sqrt{3}}{2}$ on the xy-plane.

We start with *inside* the integral, where we convert the function into cylindrical coordinates, and replace the $dzdydx$ by the volume element $rdzdrd\theta$. We now need to set up the limits.

Cylindrical coordinates essentially means one uses polar coordinates on the xy-plane: the limits for r and θ should give you the disk D (the "shadow" of the solid region). The graphs $z = 1/2$ and $z = \sqrt{1 - x^2 - y^2}$ when converted to cylindrical coordinates become $z = 1/2$ and $z = \sqrt{1 - r^2}$ respectively.

So the answer is:

$$\int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{\frac{1}{2}}^{\sqrt{1-r^2}} (r^2 + z^2) r dz dr d\theta$$

□

Example 4 (Done in class, except the calculation)**.** *Chop up the unit ball* $x^2 + y^2 + z^2 \leq 1$ *by the plane* $z = \frac{1}{\sqrt{2}}$*. What is the volume of the top portion* E *?*

Solution. The picture is similar to the one for the previous example, except that the plane is at a different height.

In class, we figured out that we can set up the triple integral in spherical coordinates as follows:

$$\iiint_E dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2}\cos\theta}}^{1} \rho^2 sin\phi d\rho d\phi d\theta$$

We do the calculation here:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\theta}}^{1} \rho^2 sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3} sin\phi\Big|_{\frac{1}{\sqrt{2}\cos\theta}}^{1} d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \Big(\frac{sin\phi}{3} - \frac{1}{6\sqrt{2}}\cdot\frac{1}{\cos^3\phi}\sin\phi\Big) d\phi d\theta$$

Doing the integral of the two terms separately (the second term by the substitution $u = \cos\phi$), we get:

$$= \int_0^{2\pi} \Big(\frac{-\cos\phi}{3}\Big|_0^{\frac{\pi}{4}} + \frac{1}{6\sqrt{2}}\cdot\frac{1}{2\cos^2\phi}\Big|_0^{\frac{\pi}{4}}\Big) d\theta = \int_0^{2\pi} \Big(-\frac{1}{3\sqrt{2}} + \frac{1}{3}\Big) + \Big(\frac{1}{6\sqrt{2}} - \frac{1}{12\sqrt{2}}\Big) d\theta$$

$$= 2\pi\Big(-\frac{1}{3\sqrt{2}} + \frac{1}{3} + \frac{1}{12\sqrt{2}}\Big) = \frac{2\pi}{3} - \frac{\pi}{2\sqrt{2}}.$$

That's the answer I promised in class! □
