Line integral strategies for Midterm 2

Math 120 Section 4

A question involving a line integral would ask you to evaluate something like:

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \langle P, Q \rangle$ is some vector field (let us stick to $2D$-vector fields here), and C is some curve, described in some way.

Sometimes the line integral looks like

$$\int_C P dx + Q dy$$

but this is just the line integral of the vector field $\vec{F} = \langle P, Q \rangle$ as above.

How do you do such an integral? There are *three* ways that you have learnt so far:

Method 1. Check if $\vec{F}$ is conservative. (Use the test $P_y = Q_x$.)

If yes: *Step 1.* Find the potential function $f(x, y)$ so that $\vec{F} = \nabla f$, so that

$$f_x = P \text{ and } f_y = Q.$$

This you can do by either guessing the potential function f (by trial and error) or by finding it by partial integration.

Step 2. Use the Fundamental Theorem of Line integrals - your line integral is just

$$\int_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)$$

where P_0 is the initial point of the path C and P_1 is the end point of C (the right hand side is zero if the curve C is closed).

If no: Try methods 2 or 3.

Method 2. Do the line integral directly:

Step 1. Parametrize the curve C as $\vec{r}(t) = \langle x(t), y(t) \rangle$ where $a \le t \le b$. Sometimes you need to break up C into "pieces" and parametrize each separately.

Step 2. Plug in to the line integral to get

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

which is an integral of a function of t that you can do the usual way (keep in mind tricks like substitution!). If your curve C was parametrized in "pieces", do this integral for each separately, and then add them up.

If this gets too complicated, try Method 3.

Method 3. Use Green's theorem to compute a double integral instead:

If C is closed, we get that

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA$$

where D is the region enclosed by C. (This assumes that C is positively oriented - if not, work with the positive orientation and then change the sign of the answer you get at the end.)

If C is not closed, first close up C by adding a curve C' (choose C' to be a simple enough, and oriented in the correct direction). C and C' together now bound a region D (we'll assume that this new composite curve formed by C and C' is oriented positively).

Green's theorem now implies that

$$\int_{C \cup C'} \vec{F} \cdot d\vec{r} = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA \tag{1}$$

On the other hand, we have that

$$\int_{C \cup C'} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r} \tag{2}$$

From (1) and (2), we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA - \int_{C'} \vec{F} \cdot d\vec{r} \tag{3}$$

So it only remains compute (separately) the two terms on the right hand side of (3):

Term 1 is a double integral, so use "Type 1" or "Type 2" integrals or polar coordinates depending on what D is.

Term 2 is a line integral (over a curve C' you have chosen) - parametrize C' and use Method 2 for this.

Once you have these, plug back in (3) to get your answer.