SUGGESTED EXERCISES - 3

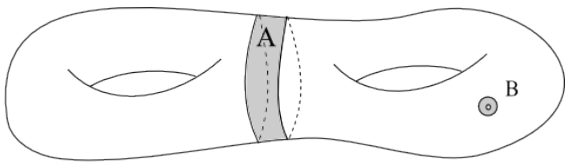
TEICHMÜLLER THEORY (MATH 191B), WINTER 2013-4

In what follows $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *is the Riemann sphere.*

(1) Let $f : \mathbb{H}^2 \to \mathbb{C}$ be the holomorphic function $f(z) = z^\alpha$ where $\alpha \in \mathbb{R}$. What is the Schwarzian derivative S_f and what is its norm $\|S_f\|$ (as we defined in class)? For which values of α is f univalent?

(2) Let $f : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}}$ be the univalent holomorphic map $f(z) = z + \frac{1}{2z}$. Write down a quasiconformal homeomorphism $\hat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that extends f, *i.e* $\hat{f} = f$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

(3) Fix a marked Riemann surface Σ_0 to be the basepoint of Teichmüller space $\mathcal{T}(\Sigma_0)$, and let A be an embedded annulus that is non-trivial in homotopy (*i.e.* cannot be shrunk to a point).



Show that there exists a sequence

$$x_n = [f_n : \Sigma_0 \to \Sigma_n] \in \mathcal{T}(\Sigma_0)$$

such that

- the Beltrami differential of each quasiconformal homeomorphism f_n is supported on A, that is, is identically zero on $\Sigma_0 \setminus A$, *and*
- the sequence x_n diverges in $\mathcal{T}(\Sigma_0)$ as $n \to \infty$.

(Difficult bonus question:) Does such a sequence exist if A is homotopically trivial and very small (like B in the figure)?

(4) Consider the set of four points $p_1 = -1, p_2 = 0, p_3 = 1, p_4 = 3$ on $\widehat{\mathbb{C}}$.

- Show that there is no conformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f(p_1) = p_1$, $f(p_2) = p_2$, $f(p_3) = p_4$ and $f(p_4) = p_3$.
- Construct a quasiconformal homeomorphism that does the "swap" as in the previous part.
- Show there exists a $K_0 > 1$ such that any quasiconformal swap as in the previous part must have quasiconformal distortion at least K_0.
- What is a set of four points on $\widehat{\mathbb{C}}$ such that the group of conformal homeomorphisms of $\widehat{\mathbb{C}}$ permuting them is the largest? Can it be the entire symmetric group S_4?