Introduction to the Calculus of Variations: Lecture 9

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Indian Institute of Science

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Outline

Direct methods in a classical problem

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- regularity questions (we shall take it up again the chapter 5)

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$$\inf \left\{ L(c) : c \in C^1\left([0, T]; \mathbb{R}^d\right), c(0) = p_1, c(T) = p_2. \right\} = m.$$

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However, here we shall not take this path and instead introduce local charts in M.

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$$\dot{c}^{\alpha}(t) = \frac{\partial f^{\alpha}}{\partial z^{i}}(\gamma(t))\,\dot{\gamma}^{i}(t) \qquad \text{for every } 1 \leq \alpha \leq d.$$

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where $g = (g_{ij})$ is a **positive definite symmetric** matrix

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But it is a quite difficult one and we need to slowly move towards it.

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But this time we have a control of the L^2 norm of the derivatives instead of the L^1 norm.

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But integrating by parts, we obtain

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By convergence of c_ν to c in C^0, the RHS above converges to

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So, using this and (1) and (2) and (3), we deduce

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