Introduction to the Calculus of Variations: Lecture 7-8

Swarnendu Sil

Department of Mathematics
Indian Institute of Science

Spring Semester 2021

Outline

Recap I

Second variation

So far we have seen that for a C^2 critical point $\bar{u}$ to be a minimizer of a functional I with C^3 Lagrangian density f, a necessary condition is that the second variation of I at $\bar{u}$ along ψ, which is explicitly given by the integral

$$\int_a^b \left[\left\langle f_{uu}\left(t, \bar{u}, \dot{\bar{u}}\right)\psi, \psi \right\rangle + 2\left\langle f_{u\xi}\left(t, \bar{u}, \dot{\bar{u}}\right)\psi, \dot{\psi} \right\rangle + \left\langle f_{\xi\xi}\left(t, \bar{u}, \dot{\bar{u}}\right)\dot{\psi}, \dot{\psi} \right\rangle \right] \mathrm{d}t,$$

must be **nonnegative** for any $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$.

We deduced that this forces the **Legendre condition**

$$f_{\xi\xi}\left(t, \bar{u}, \dot{\bar{u}}\right) \text{ is nonnegative definite} \qquad \text{for every } t \in (a,b).$$

We also saw that the second variation above being **positive** for any $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$ with $\psi \not\equiv 0$ is a sufficient condition. To obtain more easily checkable necessary and sufficient conditions, we wrote the second variation as an integral functional.

Recap II

We have rewritten the second variation as

$$J[\psi] := \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t,$$

where P is symmetric and assumed to be **positive definite**. This can be made into a 'perfect square' by adding a null Lagrangian. More precisely, we showed that we have

$$\begin{aligned} J[\psi] &= J[\psi] + \int_a^b \frac{d}{dt} \left[\langle W\psi, \psi \rangle \right] \, \mathrm{d}t \\ &= \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + 2 \left\langle W\psi, \dot{\psi} \right\rangle + \left\langle \left(Q + \dot{W} \right) \psi, \psi \right\rangle \right] \, \mathrm{d}t \\ &= \int_a^b \left| P^{\frac{1}{2}} \dot{\psi} + P^{-\frac{1}{2}} W\psi \right|^2 \, \mathrm{d}t, \end{aligned}$$

if W is a solution of the following *matrix Riccati equation*,

$$\dot{W} = -Q + WP^{-1}W.$$

Recap III

Jacobi equation

We have introduced the Jacobi equation and the notion of conjugate points.

$$\boxed{\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.}$$

Definition (Conjugate points)

Let Ψ be the matrix of N solutions of the Jacobi equation and satisfies

$$\Psi\left(a\right) = 0 \quad \text{and} \quad \dot{\Psi}\left(a\right) = \mathbb{I}_N.$$

A point $\bar{a} \in (a, b]$ is called a **conjugate to the point** a or simply a **conjugate point of** a if we have

$$\det \Psi\left(\bar{a}\right) = 0.$$

Recap IV

Jacobi theory

We have also seen that if there exists no point conjugate to a in $(a, b]$, then Ψ is invertible and

$$W = -P\dot{\Psi}\Psi^{-1}$$

solves the matrix Ricatti equation.

Now our goal is to show that this is a sufficient condition for $\bar{u}$ to be a minimizer.

Sufficient condition for a minimizer

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right),\ \alpha, \beta \in \mathbb{R}^N,$
$X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X}\left\{I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t\right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ *be a* **critical point** *of* I *such that*

- $f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)$ *is* **positive definite** *for every* $t \in [a, b]$,
- *there exists no point in* $(a, b]$ *which is* **conjugate** *to* a.

Then $\bar{u}$ *is a* **minimizer** *of* I.

Proof of sufficiency

Proof We need to show that

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

$$J[\psi] > c \int_a^b \left|\dot{\psi}\right|^2 \tag{1}$$

for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$,

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

$$J[\psi] > c \int_a^b \left|\dot{\psi}\right|^2 \qquad (1)$$

for all $\psi \in C^1([a,b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$, as this is sufficient for $\bar{u}$ to be a minimizer.

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

$$J\left[\psi\right] > c \int_a^b \left|\dot{\psi}\right|^2 \tag{1}$$

for all $\psi \in C^1\left([a, b]\right), \psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$, as this is sufficient for $\bar{u}$ to be a minimizer.

We set

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

$$J[\psi] > c \int_a^b \left|\dot{\psi}\right|^2 \qquad (1)$$

for all $\psi \in C^1([a,b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$, as this is sufficient for $\bar{u}$ to be a minimizer.

We set

$$\boxed{J_c[\psi] := J[\psi] - c \int_a^b \left|\dot{\psi}\right|^2 .}$$

Proof of sufficiency

Proof We need to show that there exists $c > 0$ such that

$$J[\psi] > c \int_a^b \left|\dot{\psi}\right|^2 \tag{1}$$

for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$, as this is sufficient for $\bar{u}$ to be a minimizer.

We set

$$\boxed{J_c[\psi] := J[\psi] - c \int_a^b \left|\dot{\psi}\right|^2 .}$$

This has the same form as $J[\psi]$ with P replaced by

$$P_c := P - c\mathbb{I}_N .$$

So the corresponding Jacobi equation is

$$\boxed{\frac{d}{dt}\left[(P - c\mathbb{I}_N)\dot{\psi}\right] = Q\psi .}$$

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$,

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite,

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to *a* for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0$,

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0, J_c$ has no point conjugate to a in $(a, b]$

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0, J_c$ has no point conjugate to a in $(a, b]$ and P_c is positive definite everywhere.

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0, J_c$ has no point conjugate to a in $(a, b]$ and P_c is positive definite everywhere. We choose and fix such a $c > 0$ for the rest of the proof.

So we can write

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0, J_c$ has no point conjugate to a in $(a, b]$ and P_c is positive definite everywhere. We choose and fix such a $c > 0$ for the rest of the proof.

So we can write

$$J_c\left[\psi\right] = \int_a^b \left|P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi\right|^2 \,\mathrm{d}t,$$

Proof of sufficiency

Now by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, as there is none for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also, since $f_{\xi\xi} = P$ is positive definite, for small enough $c > 0$, P_c must be positive definite as well.

Thus for small enough $c > 0, J_c$ has no point conjugate to a in $(a, b]$ and P_c is positive definite everywhere. We choose and fix such a $c > 0$ for the rest of the proof.

So we can write

$$J_c\left[\psi\right] = \int_a^b \left|P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi\right|^2 \,\mathrm{d}t,$$

where W is a solution of the corresponding Matrix Riccati equation as before.

Proof of sufficiency

Hence if $J_c[\psi] \not> 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi = 0 \qquad \text{for all } t \in (a,b)$$

for some $\psi \in C^1([a,b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\boxed{\dot{\psi} = -\left(P_c^{-1}W\right)\psi.}$$

Proof of sufficiency

Hence if $J_c[\psi] \not> 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi = 0 \qquad \text{for all } t \in (a,b)$$

for some $\psi \in C^1([a,b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\boxed{\dot{\psi} = -\left(P_c^{-1}W\right)\psi.}$$

Since ψ satisfies the initial condition $\psi(a) = 0$,

Proof of sufficiency

Hence if $J_c[\psi] \ngtr 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi = 0 \qquad \text{for all } t \in (a,b)$$

for some $\psi \in C^1([a,b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\boxed{\dot{\psi} = -\left(P_c^{-1}W\right)\psi.}$$

Since ψ satisfies the initial condition $\psi(a) = 0$, by uniqueness of solutions of ODE, we must have $\psi \equiv 0$.

Proof of sufficiency

Hence if $J_c[\psi] \ngtr 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}} W\psi = 0 \qquad \text{for all } t \in (a,b)$$

for some $\psi \in C^1([a,b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\boxed{\dot{\psi} = -\left(P_c^{-1}W\right)\psi.}$$

Since ψ satisfies the initial condition $\psi(a) = 0$, by uniqueness of solutions of ODE, we must have $\psi \equiv 0$.

So $J_c[\psi] > 0$ for all $\psi \in C^1([a,b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$.

Proof of sufficiency

Hence if $J_c[\psi] \not> 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}} W\psi = 0 \qquad \text{for all } t \in (a, b)$$

for some $\psi \in C^1([a, b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\boxed{\dot{\psi} = -\left(P_c^{-1} W\right)\psi.}$$

Since ψ satisfies the initial condition $\psi(a) = 0$, by uniqueness of solutions of ODE, we must have $\psi \equiv 0$.

So $J_c[\psi] > 0$ for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$.

Thus, for our choice of $c > 0$, we have

$$J[\psi] > c\int_a^b \left|\dot{\psi}\right|^2$$

for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$. This completes the proof. □

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right),\ \alpha, \beta \in \mathbb{R}^N$,
$X = \left\{ u \in C^1_{piece}\left([a,b];\mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta \right\}.$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a,b];\mathbb{R}^N\right)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)$ *is* **positive definite** *for every* $t \in [a,b]$. *Then there exists no point in* (a,b) *which is conjugate to* a.

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3([a, b] \times \mathbb{R}^N \times \mathbb{R}^N)$, $\alpha, \beta \in \mathbb{R}^N$,
$X = \{u \in C^1_{piece}([a, b]; \mathbb{R}^N) : u(a) = \alpha,\ u(b) = \beta\}$.

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2([a, b]; \mathbb{R}^N)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}(t, \bar{u}(t), \dot{\bar{u}}(t))$ *is* **positive definite** *for every* $t \in [a, b]$. *Then there exists no point in* (a, b) *which is conjugate to* a.

Question: Did we obtain a necessary and sufficient condition?

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right),\ \alpha, \beta \in \mathbb{R}^N$,
$X = \left\{u \in C^1_{piece}\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f\left(t, u(t), \dot{u}(t)\right)\, \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a,b]; \mathbb{R}^N\right)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)$ *is* **positive definite** *for every* $t \in [a,b]$. *Then there exists no point in* (a,b) *which is conjugate to* a.

Question: Did we obtain a necessary and sufficient condition?
NO!

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$,
$X = \left\{u \in C^1_{piece}\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)$ *is* **positive definite** *for every* $t \in [a, b]$*. Then there exists no point in* (a, b) *which is conjugate to* a*.*

Question: Did we obtain a necessary and sufficient condition?
NO!

- $f_{\xi\xi}$ is **positive definite** for every $t \in [a, b]$ is an explicit assumption! Not a necessary condition.

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$,
$X = \left\{u \in C^1_{piece}\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a,b]; \mathbb{R}^N\right)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)$ *is* **positive definite** *for every* $t \in [a,b]$. *Then there exists no point in* (a,b) *which is conjugate to* a.

Question: Did we obtain a necessary and sufficient condition?
NO!

- $f_{\xi\xi}$ is **positive definite** for every $t \in [a,b]$ is an explicit assumption! Not a necessary condition. Only $f_{\xi\xi}$ **nonnegative definite** everywhere in $[a,b]$ is necessary.

Jacobi's necessary condition

Now we want to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer.

Theorem

Let $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$,
$X = \left\{u \in C^1_{piece}\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X}\left\{I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t\right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a,b]; \mathbb{R}^N\right)$ *be a* **minimizer** *of* I *such that* $f_{\xi\xi}(t, \bar{u}(t), \dot{\bar{u}}(t))$ *is* **positive definite** *for every* $t \in [a,b]$. *Then there exists no point in* (a,b) *which is conjugate to* a.

Question: Did we obtain a necessary and sufficient condition?
NO!

- $f_{\xi\xi}$ is **positive definite** for every $t \in [a,b]$ is an explicit assumption! Not a necessary condition. Only $f_{\xi\xi}$ **nonnegative definite** everywhere in $[a,b]$ is necessary.
- b not being conjugate to a is needed for sufficiency, but is not necessary.

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* *be a conjugate point of* a*. Then there exists a Jacobi field* $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$.

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*.

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

$$\eta(t) = \sum_{i=1}^{N} \mu_i \psi_i(t)$$

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

$$\eta(t) = \sum_{i=1}^{N} \mu_i \psi_i(t)$$

which is not identically zero

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

$$\eta(t) = \sum_{i=1}^{N} \mu_i \psi_i(t)$$

which is not identically zero and satisfies

$$\eta(a) = 0 = \eta(a^*).$$

Jacobi fields and conjugate points

Before proving, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition

Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof.

Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

$$\eta(t) = \sum_{i=1}^{N} \mu_i \psi_i(t)$$

which is not identically zero and satisfies

$$\eta(a) = 0 = \eta(a^*).$$

Since each ψ_i solves the Jacobi equation, so does η. □

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition

If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *is a Jacobi field on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$, *then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = 0.$$

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition

If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *is a Jacobi field on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$, *then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = 0.$$

Proof.

Since $\eta(a) = 0 = \eta(a^*)$,

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition

If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *is a Jacobi field on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$, *then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \,\mathrm{d}t = 0.$$

Proof.

Since $\eta(a) = 0 = \eta(a^*)$, we can integrate by parts to obtain

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition

If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *is a Jacobi field on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$, *then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = 0.$$

Proof.

Since $\eta(a) = 0 = \eta(a^*)$, we can integrate by parts to obtain

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = \int_a^{a^*} \left\langle \left[-\frac{d}{dt}\left(P\dot{\eta}\right) + Q\eta\right], \eta \right\rangle \, \mathrm{d}t.$$

Vanishing of Jacobi fields and the quadratic forms

Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition

If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, *is a Jacobi field on* $[a, a^*]$ *such that* $\eta(a) = 0 = \eta(a^*)$, *then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = 0.$$

Proof.

Since $\eta(a) = 0 = \eta(a^*)$, we can integrate by parts to obtain

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \, \mathrm{d}t = \int_a^{a^*} \left\langle \left[-\frac{d}{dt}\left(P\dot{\eta}\right) + Q\eta\right], \eta \right\rangle \, \mathrm{d}t.$$

But the expression in the bracket vanishes as η is a Jacobi field. □

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof:

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \,\mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$,

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a, b) which is conjugate to a.

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a, b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a, b)$ is a conjugate point of a.

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a, b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a, b)$ is a conjugate point of a. Then as we have just shown, this implies that

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a, b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a, b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a, b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a, b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi\rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a,b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a,b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a,a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a,a^*]$ such that $\eta(a) = 0 = \eta(a^*)$. Now we set

$$\eta^* = \begin{cases} \eta & \text{if } t \in [a, a^*] \\ 0 & \text{if } t \in [a^*, b] \end{cases}.$$

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi \rangle \right] \, \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a,b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a,b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a,a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a,a^*]$ such that $\eta(a) = 0 = \eta(a^*)$. Now we set

$$\eta^* = \begin{cases} \eta & \text{if } t \in [a,a^*] \\ 0 & \text{if } t \in [a^*,b] \end{cases}.$$

Clearly η^* is piecewise C^1.

Proof of Jacobi's necessary condition

Now we are ready to prove Jacobi's necessary condition theorem.

Proof: This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \langle Q\psi, \psi\rangle \right] \,\mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a,b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a,b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a,a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a,a^*]$ such that $\eta(a) = 0 = \eta(a^*)$. Now we set

$$\eta^* = \begin{cases} \eta & \text{if } t \in [a,a^*] \\ 0 & \text{if } t \in [a^*,b] \end{cases}.$$

Clearly η^* is piecewise C^1. We shall prove that this actually is C^2.

Now since $J[\eta^*] = 0$

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J.

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$.

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^*

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta}^*(a^*) = 0$.

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta}^*(a^*) = 0$. By **uniqueness of solutions of ODE**,

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta}^*(a^*) = 0$. By **uniqueness of solutions of ODE**, this implies $\eta^* \equiv 0$,

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta}^*(a^*) = 0$. By **uniqueness of solutions of ODE**, this implies $\eta^* \equiv 0$, which is a contradiction. □

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a, b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta}^*$ is continuous across a^* and thus

$$\dot{\eta}^*(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta}^*(a^*) = 0$. By **uniqueness of solutions of ODE**, this implies $\eta^* \equiv 0$, which is a contradiction. □

Now we want to show some examples first.

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f\left(t,u,\xi\right)=f\left(\xi\right).}$$

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f\left(t, u, \xi\right) = f\left(\xi\right).}$$

This is the simplest case. The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f'\left(\dot{u}\right)\right] = 0, \quad \text{i.e. } f'\left(\dot{u}\right) = \text{constant.}}$$

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f\left(t, u, \xi\right) = f\left(\xi\right).}$$

This is the simplest case. The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f'\left(\dot{u}\right)\right] = 0, \quad \text{i.e. } f'\left(\dot{u}\right) = \text{constant.}}$$

Note that

$$\bar{u}\left(t\right) = \frac{\beta - \alpha}{b - a}\left(t - a\right) + \alpha \tag{2}$$

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f(t, u, \xi) = f(\xi).}$$

This is the simplest case. The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f'(\dot{u})\right] = 0, \quad \text{i.e. } f'(\dot{u}) = \text{constant.}}$$

Note that

$$\bar{u}(t) = \frac{\beta - \alpha}{b - a}(t - a) + \alpha \tag{2}$$

is a solution of the equation

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f(t, u, \xi) = f(\xi).}$$

This is the simplest case. The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f'(\dot{u})\right] = 0, \quad \text{i.e. } f'(\dot{u}) = \text{constant.}}$$

Note that

$$\bar{u}(t) = \frac{\beta - \alpha}{b - a}(t - a) + \alpha \tag{2}$$

is a solution of the equation and also satisfies the boundary conditions $\bar{u}(a) = \alpha$, $\bar{u}(b) = \beta$.

It is therefore a stationary point of I.

Examples I

We now consider several particular cases and examples that are arranged in order of increasing difficulty.

Case 1: Lagrangian depends only on the derivative

$$\boxed{f\left(t, u, \xi\right) = f\left(\xi\right).}$$

This is the simplest case. The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f'\left(\dot{u}\right)\right] = 0, \quad \text{i.e. } f'\left(\dot{u}\right) = \text{constant.}}$$

Note that

$$\bar{u}\left(t\right) = \frac{\beta - \alpha}{b - a}\left(t - a\right) + \alpha \tag{2}$$

is a solution of the equation and also satisfies the boundary conditions $\bar{u}\left(a\right) = \alpha$, $\bar{u}\left(b\right) = \beta$.

It is therefore a stationary point of I.

It is not, however, always a minimizer of $\left(P\right)$ as we shall see.

Example II

1. f is convex.

If f is convex, the above $\bar{u}$ is indeed a minimizer. From Jensen inequality, it follows that for any $u \in C^1([a,b])$ with $u(a) = \alpha,\ u(b) = \beta$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(\dot{u}(t))\ \mathrm{d}t &\geq f\left(\frac{1}{b-a}\int_a^b \dot{u}(t)\ \mathrm{d}t\right)\\ &= f\left(\frac{u(b)-u(a)}{b-a}\right)\\ &= f\left(\frac{\beta-\alpha}{b-a}\right) = f\left(\dot{\bar{u}}(t)\right)\\ &= \frac{1}{b-a}\int_a^b f\left(\dot{\bar{u}}(t)\right)\ \mathrm{d}t\end{aligned}$$

which is the claim. If f is not strictly convex, then, in general, there are other minimizers.

Example III

2. f is non-convex.

If f is non-convex, then (P) has, in general, **no solution** and therefore the above $\bar{u}$ is not necessarily a minimizer (in the particular example below it is a maximizer of the integral).

Consider

$$\boxed{f(\xi) = e^{-\xi^2}.}$$

and

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(\dot{u}(t))\,\mathrm{d}t \right\} = m$$

where

$$X = \left\{ u \in C^1([0,1]) : u(0) = u(1) = 0 \right\}.$$

We have from (2) that $\bar{u} \equiv 0$ and it is clearly a maximizer of I in the class of admissible functions X .

However (P) has no minimizer, as we now show.

Example IV

Let us show that $m = 0$. Let $\nu \in \mathbb{N}$ and define

$$u_\nu(x) = \nu\left(x - \frac{1}{2}\right)^2 - \frac{\nu}{4}$$

then $u_\nu \in X$ and

$$I(u_\nu) = \int_0^1 e^{-4\nu^2(x-1/2)^2}\,dx = \frac{1}{2\nu}\int_{-\nu}^{\nu} e^{-y^2}\,dy \to 0 \quad \text{as } \nu \to \infty.$$

Thus $m = 0$, as claimed. But clearly, no function $u \in X$ can satisfy

$$\int_0^1 e^{-(\dot{u}(t))^2}\,\mathrm{d}t = 0$$

and hence (P) has no solution.

Example V

Minimizer in C^1_{piece} are not necessarily C^1

Now we give an example to show that **minimizers in the class C^1_{piece} might not even be C^1**, thus we can not in general expect a gain of regularity.

Consider

$$\boxed{f(\xi) = \left(\xi^2 - 1\right)^2.}$$

$$\boxed{(P_{\text{piece}}) \quad \inf_{u \in X_{\text{piec}}} \left\{ I(u) = \int_0^1 f(\dot{u}(t))\, \mathrm{d}t \right\} = m_{\text{piece}}}$$

where

$$X_{\text{piece}} = \left\{ u \in C^1_{\text{piec}}([0,1]) : u(0) = u(1) = 0 \right\}.$$

Example VI

We can easily check that the **tent function**

$$v_1(t) = \begin{cases} t & \text{if } t \in [0, 1/2] \\ 1 - t & \text{if } t \in (1/2, 1] \end{cases}$$

is a minimizer since v is piecewise C^1 and satisfies $v_1(0) = v_1(1) = 0$ and $I(v_1) = 0$. Thus $m_{\text{piece}} = 0$.

Note that (P_{piece}) has a plethora of minimizers, not just one. Indeed, there are uncountably infinitely many minimizers. For example, the **one-sided double tent**

$$v_2(t) = \begin{cases} t & \text{if } t \in [0, 1/4] \\ \frac{1}{2} - t & \text{if } t \in [1/4, 1/2] \\ t - \frac{1}{2} & \text{if } t \in [1/2, 3/4] \\ 1 - t & \text{if } t \in [3/4, 1] \end{cases}$$

is also a minimizer.

Example VII

The **two-sided double tent**

$$v_3(t) = \begin{cases} t & \text{if } t \in [0, 1/4] \\ \dfrac{1}{2} - t & \text{if } x \in [1/4, 3/4] \\ t - 1 & \text{if } t \in [3/4, 1] \end{cases}$$

is another one. One can easily construct functions with multiple number of tents, one or two-sided or a combination of those. Any piecewise affine functions with slopes $+1$ or -1 which respects the boundary values is a minimizer. All of them are Lipschitz and of course C^1_{piece}, in fact C^∞_{piece}, but none of them are C^1!

Indeed, the minimization problem in C^1, i.e.

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(\dot{u}(t))\, \mathrm{d}t \right\} = m$$

where

$$X = \left\{ u \in C^1([0,1]) : u(0) = u(1) = 0 \right\},$$

admits **no** solution.

Example VIII

Let us first show that $m = 0$.

Consider the following sequence, which are just smoothed out versions of v_1 above,

$$u_\nu(t) = \begin{cases} t & \text{if } t \in \left[0, \frac{1}{2} - \frac{1}{\nu}\right] \\ -2\nu^2\left(t - \frac{1}{2}\right)^3 - 4\nu\left(t - \frac{1}{2}\right)^2 - t + 1 & \text{if } t \in \left(\frac{1}{2} - \frac{1}{\nu}, \frac{1}{2}\right] \\ 1 - t & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Note that $u_\nu \in X$ and

$$I(u_\nu) = \int_0^1 f(\dot{u_\nu}(t))\,\mathrm{d}t = \int_{\frac{1}{2}-\frac{1}{\nu}}^{\frac{1}{2}} f(\dot{u_\nu}(t))\,\mathrm{d}t \leq \frac{4}{\nu} \to 0.$$

This implies that indeed $m = 0$. But $I(u) = 0$ implies that $|\dot{u}| = 1$ almost everywhere.

But no function $u \in X$ can satisfy $|\dot{u}| = 1$, since by continuity of the derivative we should have either $\dot{u} = 1$ everywhere or $\dot{u} = -1$ everywhere, which is clearly incompatible with the boundary data.

Example IX

Also note that the Euler-Lagrange equation is

$$\frac{d}{dt}\left[\dot{u}\left(\dot{u}^2-1\right)\right]=0.$$

It has $\bar{u}\equiv 0$ as a solution. However, since $m=0$, it is not a minimizer as $I\left(0\right)=1$.

Case 2: Lagrangian depends on time and derivative

$$\boxed{f\left(t,u,\xi\right)=f\left(t,\xi\right).}$$

The Euler-Lagrange equation is

$$\boxed{\frac{d}{dt}\left[f_{\xi}\left(t,\dot{u}\right)\right]=0, \quad \text{i.e.} \quad f_{\xi}\left(t,\dot{u}\right)=\text{constant}.}$$

The equation is already harder to solve than the preceding one and, in general, it does not have a solution as simple as the last case.

Weierstrass example

Let

$$f(t,\xi) = t\xi^2.$$

Note that $\xi \mapsto f(t,\xi)$ is **convex** for every $t \in [0,1]$ and even **strictly convex** if $t \in (0,1]$. So things would have been very nice without the t-dependence. This example due to Weierstrass is among the first to point out that even t dependence can mess things up.

Consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(t, \dot{u}(t)) \, \mathrm{d}t \right\} = m$$

where

$$X = \left\{ u \in C^1([0,1]) : u(0) = 1, \ u(1) = 0 \right\}.$$

We will show that (P) has no C^1 or piecewise C^1 solution (not even in any Sobolev space).

Example XI

Weierstrass example

The Euler-Lagrange equation is

$$\frac{d}{dt}(t\dot{u}) = 0 \quad \Rightarrow \quad \dot{u} = \frac{c}{t} \quad \Rightarrow \quad u(t) = c \log t + d, \; t \in (0,1)$$

where c and d are constants. Observe first that such a u cannot satisfy simultaneously $u(0) = 1$ and $u(1) = 0$.
Let us also consider the following problem

$$(P_{\text{piece}}) \quad \inf_{u \in X_{\text{piece}}} \left\{ I(u) = \int_0^1 f(t, \dot{u}(t)) \, \mathrm{d}t \right\} = m_{\text{piece}}$$

where

$$X_{\text{piece}} = \left\{ u \in C^1_{\text{piece}}([0,1]) : u(0) = 1, \; u(1) = 0 \right\}.$$

We now prove that neither (P) nor (P_{piece}) have a minimizer.

For both cases it is sufficient to establish that $m_{\text{piece}} = m = 0$.

Example XII

Weierstrass example

Indeed if there exists a piecewise C^1 function v satisfying $I(v) = 0$, this would imply that $v' = 0$ a.e. in $(0, 1)$.

Since the function $v \in X_{\text{piece}}$, it should be continuous and $v(1)$ should be equal to 0. But then this means $v \equiv 0$, which does not verify the other boundary condition, namely $v(0) = 1$. Hence, neither (P) nor (P_{piece}) have a minimizer.

Now let $\nu \in \mathbb{N}$ and consider the sequence

$$u_\nu(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-\log t}{\log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right]. \end{cases}$$

Note that u_ν is piecewise C^1, $u_\nu(0) = 1$, $u_\nu(1) = 0$ and

$$I(u_\nu) = \frac{1}{\log \nu} \to 0 \quad \text{as } \nu \to \infty,$$

hence $m_{\text{piec}} = 0$.

Example XIII

Weierstrass example

We finally prove that $m = 0$.

Consider the following sequence

$$u_\nu(t) = \begin{cases} \frac{-\nu^2}{\log \nu} t^2 + \frac{\nu}{\log \nu} t + 1 & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-\log t}{\log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right]. \end{cases}$$

We easily have $u_\nu \in X$ and since

$$\dot{u_\nu}(t) = \begin{cases} \frac{\nu}{\log \nu}(1 - 2\nu t) & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-1}{t \log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right] \end{cases}$$

we deduce that

$$0 \leq I(u_\nu) = \frac{\nu^2}{\log^2 \nu} \int_0^{1/\nu} t(1 - 2\nu t)^2 \, \mathrm{d}t + \frac{1}{\log^2 \nu} \int_{1/\nu}^1 \frac{dt}{t} \to 0, \quad \text{as } \nu \to \infty.$$

This indeed shows that $m = 0$.

Example XIX

Minimizer in C^1 are not necessarily C^2

Our last example shows that even **minimizers in the class C^1 need not automatically have higher regularity, in particular, might not be C^2.**

Consider f which depends on all the variables t, u and ξ, given as

$$\boxed{f(t,u,\xi) = u^2(2t-\xi)^2.}$$

$$\boxed{(P) \quad \inf_{u \in X} \left\{ I(u) = \int_{-1}^{1} f(t, u(t), \dot{u}(t)) \, \mathrm{d}t \right\} = m}$$

where $X = \left\{ u \in C^1([0,1]) : u(-1) = 0, u(1) = 1 \right\}$.

One can easily check that the function

$$v(t) := \begin{cases} 0 & \text{if } t \in [-1,0] \\ t^2 & \text{if } t \in [0,1], \end{cases}$$

is a minimizer for (P) which is not C^2.

Need for weaker settings and Direct methods

Already in these rather simple examples, we saw that the regularity hypotheses we assumed throughout the chapter are often too strong.

Need for weaker settings and Direct methods

Already in these rather simple examples, we saw that the regularity hypotheses we assumed throughout the chapter are often too strong.

Indeed, this is another reason why we need to consider weaker spaces for proving existence of a minimizer and then trying to show that the minimizers, in some cases, enjoys better regularity.

Next day, we would show an illustration using still one dimensional problems (i.e. $n = 1$).

Thank you

Questions?