# Introduction to the Calculus of Variations: Lecture 6

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# Outline

# Recap I

We calculated the second variation of a functional.

**Theorem (Second Variation)**

*Let* $f = f(t, u, \xi) \in C^3\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right), \alpha, \beta \in \mathbb{R}^N,$
$X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}.$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

*For any* **minimizer** $\bar{u} \in X \cap C^2$ *for* $(P)$, *the integral*

$$\int_a^b \left[ \left\langle f_{uu}(t, \bar{u}, \dot{\bar{u}})\, \psi, \psi \right\rangle + 2\left\langle f_{u\xi}(t, \bar{u}, \dot{\bar{u}})\, \psi, \dot{\psi} \right\rangle + \left\langle f_{\xi\xi}(t, \bar{u}, \dot{\bar{u}})\, \dot{\psi}, \dot{\psi} \right\rangle \right] \mathrm{d}t$$

*is* **nonnegative** *for any* $\psi \in C_c^1\left([a, b]; \mathbb{R}^N\right).$

# Recap II



We have also rewritten the integral as

$$\int_a^b \left[ \left\langle \left[ f_{uu}\left(t, \bar{u}, \dot{\bar{u}}\right) - \frac{d}{dt} f_{u\xi}\left(t, \bar{u}, \dot{\bar{u}}\right) \right] \psi, \psi \right\rangle + \left\langle f_{\xi\xi}\left(t, \bar{u}, \dot{\bar{u}}\right) \dot{\psi}, \dot{\psi} \right\rangle \right] \, \mathrm{d}t$$

Now we want to show that for this expression to be nonnegative for every $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$, $f_{\xi\xi}$ must be nonnegative definite everywhere.

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Clearly $\psi \in C^1_c\left([a,b];\mathbb{R}^N\right)$ and plugging it, we obtain

$$\pi^2 \int_{t_0-\alpha}^{t_0+\alpha} \sin^2 \left[ \frac{2\pi (t - t_0)}{\alpha} \right] \left\langle f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) \zeta, \zeta \right\rangle \, \mathrm{d}t$$

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## Possible candidate for sufficiency

# Towards a sufficient condition





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Legendre wanted to 'complete the square' by adding a **null Lagrangian**.

## Legendre method

Let $W$ be an arbitrary differentiable symmetric matrix.

# Jacobi theory and Legendre method II

## Legendre method

Let $W$ be an arbitrary differentiable symmetric matrix. Then

$$0 = \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\,\mathrm{d}t \qquad \text{for all } \psi \text{ with } \psi(a) = \psi(b) = 0.$$

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$$\begin{aligned} J[\psi] &= J[\psi] + \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\,\mathrm{d}t \\ &= \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi}\right\rangle + 2\left\langle W\psi, \dot{\psi}\right\rangle + \left\langle \left(Q + \dot{W}\right)\psi, \psi\right\rangle\right]\mathrm{d}t \end{aligned}$$

# Jacobi theory and Legendre method II



## Legendre method

Let $W$ be an arbitrary differentiable symmetric matrix. Then

$$0 = \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\, \mathrm{d}t \qquad \text{for all } \psi \text{ with } \psi(a) = \psi(b) = 0.$$

Thus

$$\frac{d}{dt}\left[\langle W\psi, \psi\rangle\right] \text{ is a null lagrangian for any } W.$$

Hence adding such a term does not alter the value of $J[\psi]$. So we get

$$\begin{aligned} J[\psi] &= J[\psi] + \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\, \mathrm{d}t \\ &= \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi}\right\rangle + 2\left\langle W\psi, \dot{\psi}\right\rangle + \left\langle \left(Q + \dot{W}\right)\psi, \psi\right\rangle\right] \mathrm{d}t \end{aligned}$$

When can we make this a perfect square?

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# Jacobi theory and Legendre method III

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# Thank you

*Questions?*