#### Introduction to the Calculus of Variations

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#### **Classical Methods**

Classical Problem Euler-Lagrange Equations Hamiltonian formulation First integrals Symmetry and Noether's theorem Hamilton-Jacobi equations Second Variation Examples

The End

# Introduction to the Calculus of Variations: Lecture 6

Swarnendu Sil

Department of Mathematics Indian Institute of Science

Spring Semester 2021

## Outline

# **Classical Methods**

Classical Problem Euler-Lagrange Equations Hamiltonian formulation First integrals Symmetry and Noether's theorem Hamilton-Jacobi equations Second Variation Examples

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## Recap I

We calculated the second variation of a functional.

## Theorem (Second Variation)

Let 
$$f = f(t, u, \xi) \in \mathbb{C}^3([a, b] \times \mathbb{R}^N \times \mathbb{R}^N), \alpha, \beta \in \mathbb{R}^N,$$
  
 $X = \{u \in \mathbb{C}^1([a, b]; \mathbb{R}^N) : u(a) = \alpha, u(b) = \beta\}.$ 

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t)) \, \mathrm{d}t \right\} = m.$$

For any minimizer  $\bar{u} \in X \cap C^2$  for (P), the integral

$$\int_{a}^{b} \left[ \left\langle f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right)\psi,\psi\right\rangle + 2\left\langle f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\psi,\dot{\psi}\right\rangle + \left\langle f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\dot{\psi},\dot{\psi}\right\rangle \right] \,\mathrm{d}t$$

is nonnegative for any  $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$ .

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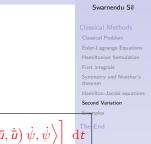
Hamilton-Jacobi equations

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We have also rewritten the integral as

$$\int_{a}^{b} \left[ \left\langle \left[ f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right) - \frac{d}{dt} f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right) \right] \psi,\psi \right\rangle + \left\langle f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right) \dot{\psi},\dot{\psi} \right\rangle \right] \, \mathrm{d}t$$

Now we want to show that for this expression to be nonnegative for every  $\psi \in C^1_c\left([a,b];\mathbb{R}^N
ight), f_{\xi\xi}$  must be nonnegative definite everywhere.

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## Lemma

If the following inequality

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holds for every  $\psi \in C_c^1([a, b]; \mathbb{R}^N)$ , then the matrix  $f_{\xi\xi}(t, \bar{u}, \dot{\bar{u}})$  is nonnegative definite

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By continuity of  $f_{\xi\xi}$ ,

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By continuity of  $f_{\xi\xi}$ , we can assume there exists  $\alpha > 0$  such that  $a < t_0 - \alpha < t_0 + \alpha < b$  and we have

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 $\left\langle f_{\xi\xi}\left(t,ar{u}\left(t
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angle <-eta$  for all  $t\in[t_{0}-lpha,t_{0}+lpha].$ 

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Choose

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$$\psi(t) = \begin{cases} \alpha \sin^2 \left[ \frac{\pi (t - t_0)}{\alpha} \right] \zeta & \text{if } t \in [t_0 - \alpha, t_0 + \alpha] \\ 0 & \text{otherwise.} \end{cases}$$

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Clearly  $\psi \in C_c^1([a, b]; \mathbb{R}^N)$ 

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Clearly  $\psi \in C^1_c\left([a,b];\mathbb{R}^N
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$$\pi^{2} \int_{t_{0}-\alpha}^{t_{0}+\alpha} \sin^{2}\left[\frac{2\pi\left(t-t_{0}\right)}{\alpha}\right] \left\langle f_{\xi\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\zeta,\zeta\right\rangle \, \mathrm{d}t$$

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$$\pi^{2} \int_{t_{0}-\alpha}^{t_{0}+\alpha} \sin^{2} \left[ \frac{2\pi \left( t - t_{0} \right)}{\alpha} \right] \left\langle f_{\xi\xi} \left( t, \bar{u} \left( t \right), \dot{\bar{u}} \left( t \right) \right) \zeta, \zeta \right\rangle \, \mathrm{d}t \\ + \alpha^{2} \int_{t_{0}-\alpha}^{t_{0}+\alpha} \sin^{4} \left[ \frac{\pi \left( t - t_{0} \right)}{\alpha} \right] \left\langle \left[ f_{uu} \left( t, \bar{u}, \dot{\bar{u}} \right) - \frac{d}{dt} f_{u\xi} \left( t, \bar{u}, \dot{\bar{u}} \right) \right] \zeta, \zeta \right\rangle \, \mathrm{d}t \geq 0$$

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But this implies

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But this implies

$$2M\alpha^3 - 2\beta\pi^2\alpha \ge 0,$$

Choose

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$$\pi^{2} \int_{t_{0}-\alpha}^{t_{0}+\alpha} \sin^{2} \left[ \frac{2\pi \left( t - t_{0} \right)}{\alpha} \right] \left\langle f_{\xi\xi} \left( t, \bar{u} \left( t \right), \dot{\bar{u}} \left( t \right) \right) \zeta, \zeta \right\rangle \, \mathrm{d}t \\ + \alpha^{2} \int_{t_{0}-\alpha}^{t_{0}+\alpha} \sin^{4} \left[ \frac{\pi \left( t - t_{0} \right)}{\alpha} \right] \left\langle \left[ f_{uu} \left( t, \bar{u}, \dot{\bar{u}} \right) - \frac{d}{dt} f_{u\xi} \left( t, \bar{u}, \dot{\bar{u}} \right) \right] \zeta, \zeta \right\rangle \, \mathrm{d}t \geq 0$$

But this implies

$$2M\alpha^3 - 2\beta\pi^2\alpha \ge 0,$$

where

$$M = \max_{t \in [t_0 - \alpha, t_0 + \alpha]} \left| \left\langle \left[ f_{uu} \left( t, \bar{u}, \dot{\bar{u}} \right) - \frac{d}{dt} f_{u\xi} \left( t, \bar{u}, \dot{\bar{u}} \right) \right] \zeta, \zeta \right\rangle \right|.$$

But this means

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But this means

$$\beta \le \frac{M}{\pi^2} \alpha^2,$$

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which we can easily contradict by letting  $\alpha \rightarrow 0$ . So we deduce

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ight)\zeta,\zeta
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for all  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^N$  and for all  $x \in \Omega$ .

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for all  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^N$  and for all  $x \in \Omega$ . This is weaker than the Legendre condition in that case

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 $\langle f_{\xi\xi}(x, \bar{u}(x), D\bar{u}(x)) a \otimes b, a \otimes b \rangle \geq 0$ 

for all  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^N$  and for all  $x \in \Omega$ . This is weaker than the Legendre condition in that case ( convexity only along rank one matrices ).

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# Towards a sufficient condition

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Can  $f_{\xi\xi} \ge 0$  be a sufficient condition? Clearly not!

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### Understanding the trouble

The reason is that the condition is purely local,

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The reason is that the condition is purely local, whereas being a minimizer is not really a local property. We go back to geodesics.

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Can  $f_{\xi\xi} \ge 0$  be a sufficient condition? Clearly not! Think of  $f(x) = x^3$ . x = 0 is not a minima! Can  $f_{\xi\xi} > 0$ , i.e. positive definite instead of nonnegative definite, be a sufficient condition?

Somewhat surprisingly, the answer is still No!

### Understanding the trouble

The reason is that the condition is purely local, whereas being a minimizer is not really a local property. We go back to geodesics. Think of the unit sphere in  $\mathbb{R}^3$  centered at the origin and consider the points A = (1,0,0), B = (0,1,0) and  $C = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ . All three points lie on the circle  $\{(x,y,0): x^2 + y^2 = 1\}$ , which being a great circle is a geodesic on the sphere. Now, the part of the circle going from A to B is a minimizing path and so is the part of the circle going from A to C can not be minimizing, as the part of the circle going from C to A is definitely shorter.

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# Symmetry and Noether's $J[\psi] := \int_{a}^{b} \left[ \left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \left\langle Q\psi, \psi \right\rangle \right] \, \mathrm{d}t, \qquad \psi \in C^{1}, \psi(\mathbf{a}) = \psi(\mathbf{b}) = \underbrace{\mathbf{0}_{\mathsf{daraulton-Jacobi equations}}_{\mathsf{Second Variation}}$ Examples

We now consider the second variation itself as an integral functional

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Note that if

$$J\left[\psi
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$$J[\psi] := \int_{a}^{b} \left[ \left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \left\langle Q\psi, \psi \right\rangle \right] \, \mathrm{d}t, \qquad \psi \in C^{1}, \psi(a) = \psi(b) = \underbrace{\mathsf{O}_{\mathsf{d}arefitor-lacobic equations}^{\mathsf{Symphetry and Noether theorem}}_{\mathsf{Complexity}} \left[ \mathsf{C}_{\mathsf{d}arefitor-lacobic equation}^{\mathsf{Symphetry and Noether theorem}} \right]$$

Note that if

$$J[\psi] > c \int_{a}^{b} \left| \dot{\psi} \right|^{2}, \qquad ext{for all } \psi \in C^{1}, \psi 
eq 0 ext{ with } \psi(a) = \psi(b) = 0,$$

for some c > 0, then  $\bar{u}$  is a minimizer. (Check!)

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Legendre wanted to 'complete the square'

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Legendre wanted to 'complete the square' by adding a **null** Lagrangian.

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Let W be an arbitrary differentiable symmetric matrix. Then

$$0 = \int_{a}^{b} \frac{d}{dt} \left[ \langle W\psi, \psi \rangle \right] \, \mathrm{d}t \qquad \text{for all } \psi \text{ with } \psi \left( a \right) = \psi \left( b \right) = 0.$$

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Thus

$$rac{d}{dt}\left[\langle W\psi,\psi
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 is a null lagrangian for any  $W$ 

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Thus

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Hence adding such a term does not alter the value of  $J[\psi]$  .

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$$J[\psi] = J[\psi] + \int_{a}^{b} \frac{d}{dt} \left[ \langle W\psi, \psi \rangle \right] \, \mathrm{d}t$$

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$$\begin{split} J[\psi] &= J[\psi] + \int_{a}^{b} \frac{d}{dt} \left[ \langle W\psi, \psi \rangle \right] \, \mathrm{d}t \\ &= \int_{a}^{b} \left[ \left\langle P\dot{\psi}, \dot{\psi} \right\rangle + 2 \left\langle W\psi, \dot{\psi} \right\rangle + \left\langle \left( Q + \dot{W} \right) \psi, \psi \right\rangle \right] \, \mathrm{d}t \end{split}$$

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When can we make this a perfect square?

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# Proposition

Suppose W is a solution of the following matrix Riccati equation,

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# Proposition

Suppose W is a solution of the following matrix Riccati equation,

 $\dot{W} = -Q + WP^{-1}W.$ 

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Note that since P is symmetric and positive definite,

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Note that since P is symmetric and positive definite,  $P^{\frac{1}{2}}$  is well defined

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$$\dot{W} = -Q + WP^{-1}W,$$

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$$\dot{W} = -Q + WP^{-1}W,$$

let us substitute

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$$W = -P \dot{\Psi} \Psi^{-1}.$$

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$$W = -P\dot{\Psi}\Psi^{-1}.$$

Plugging it in the Riccati equation, we obtain

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$$\frac{d}{dt}\left(P\dot{\Psi}\right)=Q\Psi.$$

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Any solution  $\varPsi$  of the above equation

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Any solution  $\varPsi$  of the above equation would furnish a solution W of the Riccati equation

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Plugging it in the Riccati equation, we obtain

$$\frac{d}{dt}\left(P\dot{\Psi}\right)=Q\Psi.$$

Any solution  $\Psi$  of the above equation would furnish a solution W of the Riccati equation if  $\Psi$  is **invertible.** 

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However, the equation above has another nice interpretation.

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$$J[\psi] := \int_{a}^{b} \left[ \left\langle P\dot{\psi}, \dot{\psi} \right\rangle + \left\langle Q\psi, \psi \right\rangle \right] \, \mathrm{d}t, \qquad \psi \in C^{1}, \psi(a) = \psi(b) = 0^{\text{Hensiton-Jacobi equation}}_{\text{The End}}$$

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Hamilton-Jacobi equations

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This is called the **Jacobi equation** and its solutions (for a given *u*) is called a **Jacobi field along** *u*. Before proceeding further, we need the notion of conjugate points.

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Hamilton-Jacobi equations

### Definition

Let  $\Psi$  be the matrix of N solutions of the Jacobi equation,

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Hopefully, by now all of you can see the point. If there are no interior conjugate points to a, then  $\Psi$  would be invertible and would furnish a solution to the Riccati equation.

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# **Thank you** *Questions?*