Introduction to the Calculus of Variations: Lecture 5

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Spring Semester 2021

Outline

Hamilton-Jacobi equations

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$$\frac{\partial S}{\partial \alpha_i} \quad \text{is a first integral of Hamilton's equations for each } 1 \leq i \leq m.$$

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constitute a general solution of the Hamilton's equations (**H**).

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$$0 = \frac{d}{dt}\left(\frac{\partial}{\partial\alpha_i} S\right) = \left\langle \dot{u} - H_v, \frac{\partial}{\partial u}\left[\frac{\partial S}{\partial\alpha_i}\right]\right\rangle \text{ for } 1 \leq i \leq N.$$

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This completes the proof. □

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On the other hand, (**H**) is nothing but the characteristic system for the nonlinear first order PDE (**HJE**).

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Theorem (Second Variation)

Let $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ *be given and* $X = \left\{u \in C^1\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Consider the problem*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

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If $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ is a minimizer for (P), then the following integral

$$\int_a^b \left[\left\langle f_{uu}(t, \bar{u}, \dot{\bar{u}})\, \psi, \psi \right\rangle + 2 \left\langle f_{u\xi}(t, \bar{u}, \dot{\bar{u}})\, \psi, \dot{\psi} \right\rangle + \left\langle f_{\xi\xi}(t, \bar{u}, \dot{\bar{u}})\, \dot{\psi}, \dot{\psi} \right\rangle \right]$$

is **nonnegative** *for any $\psi \in C_c^1\left([a, b]; \mathbb{R}^N\right)$.*

Calculating the second variation

Proof.

As we did in deriving the EL equations, we take $\psi \in C_c^1\left([a, b]; \mathbb{R}^N\right)$. Thus for any $h \in \mathbb{R}$, we have $\bar{u} + h\psi \in X$.

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The rest is a straight forward calculation. □

To understand the expression better, we integrate by parts in the mixed term to arrive at

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$$\int_a^b \left\langle f_{u\xi}\left(t, \bar{u}, \dot{\bar{u}}\right)\psi, \dot{\psi}\right\rangle = -\int_a^b \left\langle \frac{d}{dt}\left[f_{u\xi}\left(t, \bar{u}, \dot{\bar{u}}\right)\right]\psi, \psi\right\rangle$$

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The heuristic argument is, since ψ vanishes at the boundary, we have a Poincaré inequality. Roughly, the function itself can not be large while keeping its derivative small. But the converse is quite possible! The function can be small with large derivative! Why?

Quadratic functional related to second variation

In view of the integration by parts that we just did, we can rewrite the expression as

$$\int_a^b \left[\left\langle \left[f_{uu}\left(t, \bar{u}, \dot{\bar{u}}\right) - \frac{d}{dt} f_{u\xi}\left(t, \bar{u}, \dot{\bar{u}}\right) \right] \psi, \psi \right\rangle + \left\langle f_{\xi\xi}\left(t, \bar{u}, \dot{\bar{u}}\right) \dot{\psi}, \dot{\psi} \right\rangle \right].$$

The important point here is that the matrix $f_{\xi\xi}$ plays the dominant role here in determining whether the quadratic form will be nonnegative or not.

The heuristic argument is, since ψ vanishes at the boundary, we have a Poincaré inequality. Roughly, the function itself can not be large while keeping its derivative small. But the converse is quite possible! The function can be small with large derivative! Why? It can oscillate a lot!

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We now formalize the heuristic argument.

Lemma

If for every $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$, *we have the following inequality*

$$\int_a^b \left[\left\langle \left[f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right) - \frac{d}{dt} f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right) \right] \psi, \psi \right\rangle + \left\langle f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right) \dot{\psi}, \dot{\psi} \right\rangle \right] \geq 0,$$

then the matrix $f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right)$ *is nonnegative definite for every* $\left(t,\bar{u},\dot{\bar{u}}\right)$.

Thank you

Questions?