Introduction to the Calculus of Variations: Lecture 4

Swarnendu Sil

Department of Mathematics
Indian Institute of Science

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Outline

Symmetry and conserved quantities: Noether's theorem

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I spent half a slide for a reason, to remind you something that I sincerely hope should not need reminding – that mathematicians do not have to be men and genius does not have a gender.

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Thus, the curve $u : [a, b] \to \mathbb{R}^N$ is transformed to $u \circ \phi_\tau^{-1} : [a + \tau, b + \tau] \to \mathbb{R}^N$.

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$$\int_a^\theta f\left(u(t), \dot{u}(t)\right) \, \mathrm{d}t = \int_{a+\tau}^{\theta+\tau} f\left(u \circ \phi_\tau^{-1}(s), \frac{d}{ds}\left[u \circ \phi_\tau^{-1}\right](s)\right) \mathrm{d}s.$$

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Thus, we have the invariance

$$I_\theta[u] = I_\theta[u \circ \phi_\tau^{-1}] \qquad \text{for any } \theta \in (a, b].$$

Definition (Invariance)

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$$\left\langle f_\xi\left(t, u\left(t\right), \dot{u}\left(t\right)\right), \left.\frac{d}{ds}\left[\bar{\phi}_s\left(u(t)\right)\right]\right|_{s=0}\right\rangle \\ + \left[f\left(t, u\left(t\right), \dot{u}\left(t\right)\right) - \left\langle f_\xi\left(t, u\left(t\right), \dot{u}\left(t\right)\right), \dot{u}\left(t\right)\right\rangle\right] \left.\frac{d}{ds}\left[\phi_s^0\left(t\right)\right]\right|_{s=0}$$

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In the last line we substituted for f_u using the EL equations.

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Since $\theta \in (a, b]$ is arbitrary, this proves the special case.

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This proves the theorem. $\square$

Examples of conservation laws

Now we illustrate Noether's theorem by some simple examples.

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Example 1: Linear momentum conservation Let $m > 0$ be the mass and $x(t) \in \mathbb{R}^3$ be the position of a point particle. Let the potential energy function $U : \mathbb{R}^3 \to \mathbb{R}$ be independent of the x_3 coordinate, i.e.

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$$p_3 := m \dot{x}_3$$

is a first integral. Similarly, if $U \equiv 0$, then $m\dot{x}$ is a first integral.

Examples of conservation laws

Example 2: Angular momentum conservation Once again consider the same Lagrangian density.

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$$M_3 := \left(u \wedge m\dot{x}\right) \cdot e_3$$

is a first integral, where $\wedge$ is the vector product (cross product).

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See assignments and notes for more on this.

Thank you

Questions?