Introduction to the Calculus of Variations: Lecture 3

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Spring Semester 2021

Outline

Hamiltonian formulation

So far we have investigated the stationary points of the following functional

$$u \mapsto I(u) := \int_a^b f\left(t, u\left(t\right), \dot{u}\left(t\right)\right)\, \mathrm{d}t, \qquad \text{for } u \in X.$$

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$$H\left(x,u,v\right) = \sup_{\xi \in \mathbb{R}^N} \left\{\left\langle v, \xi\right\rangle - f\left(x,u,\xi\right)\right\}.$$

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$$x^* = \nabla f(x) \quad \text{and} \quad x = \nabla f^*(x^*).$$

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So plugging it in the definition, we deduce

$$f^*(x^*) = \frac{1}{p'}|x^*|^{p'}.$$

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The Euler-Lagrange equations for the functional

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and $\quad v = f_\xi(t,u,\xi) \quad$ *if and only if* $\quad \xi = H_v(t,u,v)$.

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Note that the coercivity assumptions imply

$$\lim_{|\xi|\to\infty} \frac{f\left(t, u, \xi\right)}{|\xi|} = +\infty \qquad \text{for every } (t, u) \in [a, b] \times \mathbb{R}^N.$$

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Now continuity of H follows from the continuity of $(t,u) \mapsto f(t,u,\xi)$ for every ξ.

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The last equation also proves $\xi = H_v$ if and only if $v = f_{\xi}$.

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Now we are ready to state and prove our main theorem regarding the Hamiltonian formulation.

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$$\frac{d}{dt} f_\xi = \dot{v} = -H_u = -\langle v - f_\xi, \xi_u \rangle + f_u = f_u,$$

which is the (**EL**) equations. Conversely, if $u \in C^2\left([a,b]; \mathbb{R}^N\right)$ satisfies (**EL**) then

$$v(t) = f_\xi\left(t, u(t), \dot{u}(t)\right) \quad \text{implies } \dot{u} = H_v.$$

Also,

$$\dot{v} = \frac{d}{dt} f_\xi = f_u = \langle v - f_\xi, \xi_u \rangle - H_u = -H_u,$$

verifying (**H**). □

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vanishes identically.

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Proof.

Along each integral curve $(u(t), v(t))$ of the Hamilton's equations, we have

$$\dot{\Phi}\left(t\right) = \frac{d}{dt}\Phi = \left\langle \Phi_u, \dot{u} \right\rangle + \left\langle \Phi_v, \dot{v} \right\rangle = \left\{\Phi, H\right\}.$$

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This however, is not a coincidence! This is just a special instance of a profound general fact known as Noether's theorem.

Thank you

Questions?