Introduction to the Calculus of Variations: Lecture 21

Swarnendu Sil

Department of Mathematics
Indian Institute of Science

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Outline

Regularity

- Regularity questions in the Calculus of variations
- L^2 regularity
  - Regularity for harmonic functions
  - Interior L^2 estimate for elliptic systems
  - Hole filling technique

Caccioppoli inequality for elliptic systems

Theorem (Caccioppoli inequality for elliptic systems)

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for all $\zeta \in \mathbb{R}^N$, for some constant $c = c\left(\lambda, \left\|A\right\|_{L^\infty}\right) > 0$.

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This completes the proof. □

L^2 regularity

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Proof. Fix $x_0 \in \Omega$ and $0 < 2R < \operatorname{dist}\left(x_0, \partial\Omega\right)$.

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where $f \in L^2\left(\Omega; \mathbb{R}^N\right)$. Then $u \in W^{2,2}_{loc}\left(\Omega; \mathbb{R}^N\right)$ and for any $\widetilde{\Omega} \subset\subset \Omega$, we have the estimate

$$\left\|\nabla^2 u\right\|_{L^2(\widetilde{\Omega})} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right),$$

where $c > 0$ is a constant depending only on $\widetilde{\Omega}$ and Ω.

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Now we are going to show another interesting corollary of the Caccioppoli inequality.

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Let $u \in W^{1,2}_{loc}\left(\mathbb{R}^2; \mathbb{R}^N\right)$ be a weak solution of

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Hole filling technique

Since $\frac{c}{c+1} < 1$, the last one is a decay estimate. Then for any $0 < \rho < R$, we have by interpolating,

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, \mathrm{d}x \leq 2^\alpha \left(\frac{\rho}{R}\right)^\alpha \int_{B_R(x_0)} |\nabla u|^2 \, \mathrm{d}x,$$

where $\alpha := \log_2\left(\frac{c+1}{c}\right)$. This proves the result. □

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Letting $R \to \infty$, we obtain the conclusion. □

Thank you

Questions?