Introduction to the Calculus of Variations: Lecture 20

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Outline

Quasiconvexity of the determinant

Proposition

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$$\det(\xi + \nabla\phi) = \det \begin{pmatrix} \xi_{11} + \dfrac{\partial \phi^1}{\partial x_1} & \xi_{12} + \dfrac{\partial \phi^1}{\partial x_2} \\ \xi_{21} + \dfrac{\partial \phi^2}{\partial x_1} & \xi_{22} + \dfrac{\partial \phi^2}{\partial x_2} \end{pmatrix}$$

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$$= (\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) + \left(\xi_{11} \frac{\partial \phi^2}{\partial x_2} + \xi_{22} \frac{\partial \phi^1}{\partial x_1} - \xi_{12} \frac{\partial \phi^2}{\partial x_1} - \xi_{21} \frac{\partial \phi^1}{\partial x_2} \right)$$

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&= (\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) + \left(\xi_{11}\frac{\partial\phi^2}{\partial x_2} + \xi_{22}\frac{\partial\phi^1}{\partial x_1} - \xi_{12}\frac{\partial\phi^2}{\partial x_1} - \xi_{21}\frac{\partial\phi^1}{\partial x_2}\right) \\
&= \det\xi + \left(\xi_{11}\frac{\partial\phi^2}{\partial x_2} + \xi_{22}\frac{\partial\phi^1}{\partial x_1} - \xi_{12}\frac{\partial\phi^2}{\partial x_1} - \xi_{21}\frac{\partial\phi^1}{\partial x_2}\right).
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We have already proved the weak continuity of determinants. Using that result and the Mazur lemma, we can prove the weak lower semicontinuity of functionals with polyconvex integrands.

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The integrand need **not be convex** as a function of the gradient variable. Also, if $u_s \rightharpoonup u$ in $W^{1,p}$ for some $2 < p < \infty$, both the convergences in the hypothesis above are satisfied and consequently the theorem holds.

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$$f(\xi, \theta) \geq \begin{cases} c_1 |\xi|^2 + c_2 |\theta|^q & \text{if } p = 2, \\ c_1 |\xi|^p & \text{if } p > 2, \end{cases} \qquad \text{for all } \xi \in \mathbb{R}^{2\times 2}, \theta \in \mathbb{R},$$

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for some $c_1, c_2 > 0$, some exponent $q > 1$. Let

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Let $2 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^2$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^2\right)$ be given. Let $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $F = F(\xi,\theta)$ be continuous, convex and satisfies,

$$f(\xi,\theta) \geq \begin{cases} c_1 |\xi|^2 + c_2 |\theta|^q & \text{if } p = 2, \\ c_1 |\xi|^p & \text{if } p > 2, \end{cases} \qquad \text{for all } \xi \in \mathbb{R}^{2\times 2}, \theta \in \mathbb{R},$$

for some $c_1, c_2 > 0$, some exponent $q > 1$. Let

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Now we can use the wlsc theorem to conclude. □

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It is perfectly possible for the sum of two functions, none of which are L^2, to be square integrable.

Things are far from clear when $n \geq 2$. For example, if $n = 2$,

$$\Delta u\left(x, y\right) = \frac{\partial^2 u}{\partial x^2}\left(x, y\right) + \frac{\partial^2 u}{\partial y^2}\left(x, y\right).$$

So first of all, all second derivatives of u does not even appear in the equation, only the pure ones do.

Secondly, it is not even clear if we can conclude that the pure second derivatives are in L^2.

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However, this somewhat miraculous conclusion is actually true in all dimensions.

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Let $u \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ be a weak solution of the following

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The main tool is an inequality called the **Caccioppoli inequality** or the **reverse Poincaré inequality**. We begin with the simplest case, the case of harmonic functions.

Theorem (Caccioppoli inequality)

Let $u \in W^{1,2}(\Omega)$ *be a weak solution of* $\Delta u = 0$, *i.e.*

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for some universal constant $c > 0$.

The regularity is a consequence of the competition between reverse Poincaré and the usual Poincaré-Sobolev inequalities.

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Thank you

Questions?