Introduction to the Calculus of Variations: Lecture 2

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Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Chapter 2: Classical Methods

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Outline

Classical Methods
- Classical Problem
- Euler-Lagrange Equations
  - Fundamental Lemma of Calculus of Variations
  - Derivation of EL equations
  - Constrained Minimization: Lagrange Multipliers
  - Example: Newton's laws for a point particle
- Hamiltonian formulation
- First integrals
- Symmetry and Noether's theorem
- Hamilton-Jacobi theory and Optimal control
- Second Variation
- Examples

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

The classical problems and methods were all concerned with the case $n = 1$. The case $n \geq 2$, the so-called **multiple integrals in the calculus of variations** is much harder and it took time to develop the tools needed to address it.

The problem with prescribed Dirichlet boundary value

Let $a, b \in \mathbb{R}$, $a < b$ and $\alpha, \beta \in \mathbb{R}^N$ be given. Let $f \in C^2\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$ and $X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. Consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f\left(t, u(t), \dot{u}(t)\right)\, \mathrm{d}t \right\} = m.$$

The Dirichlet condition is encoded in the choice of the space X. Other choices for the space are possible, which we shall discuss later.

Euler-Lagrange Equations

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Theorem (Euler-Lagrange equation)

Let $f = f(t, u, \xi) \in C^2\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ *be given and* $X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Consider the problem*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Euler-Lagrange Equations

Theorem (Euler-Lagrange equation)

Let $f = f(t, u, \xi) \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ *be given and* $X = \left\{u \in C^1\left([a,b];\mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Consider the problem*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

If $\bar{u} \in X \cap C^2\left([a,b];\mathbb{R}^N\right)$ *is a minimizer for* (P),

Theorem (Euler-Lagrange equation)

Let $f = f(t, u, \xi) \in C^2\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ *be given and* $X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. *Consider the problem*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\, \mathrm{d}t \right\} = m.$$

If $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ *is a minimizer for* (P), *then* $\bar{u}$ *satisfies,*

$$\textbf{(EL)} \quad \frac{d}{dt}\left[f_\xi\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)\right] = f_u\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right), \quad \text{for every } t \in (a, b).$$

Introduction to the Calculus of Variations
Swarnendu Sil
Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Theorem (Euler-Lagrange equation)

Let $f = f(t, u, \xi) \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ be given and $X = \left\{u \in C^1\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. Consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f\left(t, u(t), \dot{u}(t)\right)\, \mathrm{d}t \right\} = m.$$

If $\bar{u} \in X \cap C^2\left([a,b]; \mathbb{R}^N\right)$ is a minimizer for (P), then $\bar{u}$ satisfies,

$$(\mathbf{EL}) \quad \frac{d}{dt}\left[f_\xi\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)\right] = f_u\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right), \quad \text{for every } t \in (a,b).$$

Thus the **Euler-Lagrange** *equations are a system of N second order ODEs.*

$$f_{\xi\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) \ddot{\bar{u}}(t) + f_{u\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) \dot{\bar{u}}(t) + f_{t\xi}\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) - f_u\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) = 0.$$

Fundamental Lemma of Calculus of Variations

To derive the Euler-Lagrange equations, we first need a lemma, called the fundamental lemma of Calculus of Variations.

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Fundamental Lemma of Calculus of Variations

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

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Let $\Omega \subset \mathbb{R}^n$ open, $1 \leq p \leq \infty$. We say $u \in L^p_{\text{loc}}(\Omega)$ if $u \in L^p(K)$ for every $K \subset \Omega$ compact.

Theorem (Fundamental lemma of the calculus of variations)

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}\left(\Omega; \mathbb{R}^N\right)$ be such that

$$\int_\Omega \langle u(x), \psi(x) \rangle \, \mathrm{d}x = 0, \qquad \textit{for every } \psi \in C_c^\infty\left(\Omega; \mathbb{R}^N\right) \qquad (1)$$

then $u = 0$ a.e. in Ω.

See Lecture notes and Assignments for some corollaries of this lemma.

Proof of Fundamental Lemma of Calculus of Variations.

Enough to prove for $N = 1$. (why?) Pick $K \subset \Omega$ be compact arbitrarily. It is enough to show $u = 0$ a.e. in K. Set

$$v := \begin{cases} \operatorname{sgn} u & \text{if } x \in K, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus K. \end{cases}$$

Mollify to find a sequence $\{v_s\} \subset C_c^\infty(\Omega)$ such that $\|v_s\|_{L^\infty} \leq \|v\|_{L^\infty}$ and $v_s \to v$ in L^1. Then, up to a subsequence $v_s \to v$ a.e.

Now, since $v_s \in C_c^\infty(\Omega)$, for every s, we have

$$\int_\Omega u(x)\, v_s(x)\, \mathrm{d}x = 0 \qquad \text{for every } s \geq 1.$$

By dominated convergence theorem, we have

$$\int_\Omega u(x)\, v(x)\, \mathrm{d}x = 0 \quad \Rightarrow \int_K |u|\, \mathrm{d}x = 0 \quad \Rightarrow u = 0 \text{ a.e. in } K.$$

□

Introduction to the Calculus of Variations
Swarnendu Sil
Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Proof of theorem about EL equations.

Take $\phi \in C_c^1\left((a,b);\mathbb{R}^N\right)$. Thus $\phi(a) = 0 = \phi(b)$

Derivation of the Euler-Lagrange Equations

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Proof of theorem about EL equations.

Take $\phi \in C_c^1\left((a,b)\,;\mathbb{R}^N\right)$. Thus $\phi\,(a) = 0 = \phi\,(b)$ and for any $h \in \mathbb{R}$, we have $\bar{u} + h\phi \in X$.

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Derivation of the Euler-Lagrange Equations

Proof of theorem about EL equations.

Take $\phi \in C_c^1\left((a,b);\mathbb{R}^N\right)$. Thus $\phi(a) = 0 = \phi(b)$ and for any $h \in \mathbb{R}$, we have $\bar{u} + h\phi \in X$. Now we define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(h) := I\left(\bar{u} + h\phi\right)$.

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

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Then $g \in C^1(\mathbb{R})$ (Check!)

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Proof of theorem about EL equations.

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Then $g \in C^1(\mathbb{R})$ (Check!) and since $\bar{u}$ is a minimizer, g must have a local minima at 0. Thus we must have $g'(0) = 0$.

Derivation of the Euler-Lagrange Equations

Proof of theorem about EL equations.

Take $\phi \in C_c^1\left((a,b);\mathbb{R}^N\right)$. Thus $\phi(a) = 0 = \phi(b)$ and for any $h \in \mathbb{R}$, we have $\bar{u} + h\phi \in X$. Now we define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(h) := I(\bar{u} + h\phi)$.

Then $g \in C^1(\mathbb{R})$ (Check!) and since $\bar{u}$ is a minimizer, g must have a local minima at 0. Thus we must have $g'(0) = 0$. So we compute (Check!)

$$\begin{aligned} 0 &= \frac{d}{dh}\left[I(\bar{u} + h\phi)\right]\Big|_{h=0} \\ &= \int_a^b \left[\left\langle f_\xi\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right), \dot{\phi}(t)\right\rangle + \left\langle f_u\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right), \phi(t)\right\rangle\right] \mathrm{d}t \\ &= \int_a^b \left\langle \left[-\frac{d}{dt} f_\xi\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right) + f_u\left(t, \bar{u}(t), \dot{\bar{u}}(t)\right)\right], \phi(t)\right\rangle \mathrm{d}t, \end{aligned}$$

where we have used integration by parts in the last line and the fact that $\phi(a) = 0 = \phi(b)$. Conclude by applying the fundamental lemma. □

Remarks

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

- The whole point of the derivation of the Euler-Lagrange equation is the proof and the calculations leading to the derivation, not the equations themselves.

  As we shall see in a moment, different problems might lead to different equations, but **the general method is the same**.

- The Euler-Lagrange equation is sometimes called the **first variation formula**. The name comes from the fact that given a functional $I(u)$, its **first variation** at $\bar{u}$ is (its Gateaux derivative at $\bar{u}$,)

$$\delta I\left(\bar{u}, \phi\right) := \left.\frac{d}{dh}\left[I\left(\bar{u} + h\phi\right)\right]\right|_{h=0}.$$

  The EL equation is just follows from "first variation = 0."

  This is also the reason for the name Calculus of variations. All we used to do is to compute the first variation and the second variation!

Constrained Minimization: Lagrange Multipliers I

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Now we are going to derive the Euler-Lagrange equation for problems with additional constraints.

Theorem (Lagrange Multiplier)

Let $f = f(t, u, \xi) \in C^2\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $g = g(t, u, \xi) \in C^2\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ *be given and*

$$X = \left\{ \begin{array}{l} u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta, \\ \hfill \displaystyle\int_a^b g\left(t, u(t), \dot{u}(t)\right) \,\mathrm{d}t = 0 \end{array} \right\}.$$

Now consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f\left(t, u(t), \dot{u}(t)\right) \,\mathrm{d}t \right\} = m.$$

Constrained Minimization: Lagrange Multipliers II

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Theorem (Lagrange Multiplier)

[continued..]

If $\bar{u} \in X \cap C^2\left([a,b];\mathbb{R}^N\right)$ is a minimizer for (P), then there exists a $\lambda \in \mathbb{R}$, called the **Lagrange multiplier**, *such that $\bar{u}$ satisfies,*

$$\left(\frac{d}{dt}\left[f_\xi\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)\right] - f_u\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)\right) = \lambda\left(\frac{d}{dt}\left[g_\xi\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)\right] - g_u\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)\right)$$

for every $t \in (a,b)$, provided

$$\frac{d}{dt}\left[g_\xi\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)\right] \not\equiv g_u\left(t,\bar{u}(t),\dot{\bar{u}}(t)\right)$$

in (a,b).

Remarks on Lagrange Multiplier theorem

- Sometimes the equation is written as (parameter is $-\lambda$)

$$\frac{d}{dt}\left[f_{\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)+\lambda g_{\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\right]$$
$$=f_{u}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)+\lambda g_{u}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right).$$

- The theorem basically says: The **constrained minimization** of

$$\int_a^b f\left(t,u\left(t\right),\dot{u}\left(t\right)\right)\,\mathrm{d}t$$

with the constraint

$$\int_a^b g\left(t,u\left(t\right),\dot{u}\left(t\right)\right)\,\mathrm{d}t=0,$$

is equivalent to the **unconstrained minimization** of

$$\int_a^b f\left(t,u\left(t\right),\dot{u}\left(t\right)\right)\,\mathrm{d}t+\lambda\int_a^b g\left(t,u\left(t\right),\dot{u}\left(t\right)\right)\,\mathrm{d}t$$

for some $\lambda\in\mathbb{R}$.

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Proof of the Lagrange Multiplier theorem I

Proof of the Lagrange multiplier theorem.

Since

$$\frac{d}{dt}\left[g_{\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\right]-g_{u}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\not\equiv 0$$

in (a,b), the fundamental lemma implies the existence of $\psi\in C_c^{\infty}\left((a,b);\mathbb{R}^N\right)$ such that

$$\int_a^b\left[\left\langle g_{\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right),\dot{\psi}\left(t\right)\right\rangle+\left\langle g_{u}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right),\psi\left(t\right)\right\rangle\right]\mathrm{d}t\neq 0.$$

Now we can normalize if necessary to obtain $\psi\in C_c^{\infty}\left((a,b);\mathbb{R}^N\right)$ such that

$$\int_a^b\left[\left\langle g_{\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right),\dot{\psi}\left(t\right)\right\rangle+\left\langle g_{u}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right),\psi\left(t\right)\right\rangle\right]\mathrm{d}t=1.$$

Proof of the Lagrange Multiplier theorem II

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Now we pick $\phi \in C_c^\infty\left((a,b);\mathbb{R}^N\right)$ arbitrary, ψ as above and for $\varepsilon, h \in \mathbb{R}$, we set

$$F(\varepsilon, h) := \int_a^b f\left(t, \bar{u}(t) + \varepsilon\phi(t) + h\psi(t), \dot{\bar{u}}(t) + \varepsilon\dot{\phi}(t) + h\dot{\psi}(t)\right)\,\mathrm{d}t$$

$$G(\varepsilon, h) := \int_a^b g\left(t, \bar{u}(t) + \varepsilon\phi(t) + h\psi(t), \dot{\bar{u}}(t) + \varepsilon\dot{\phi}(t) + h\dot{\psi}(t)\right)\,\mathrm{d}t$$

So $G \in C^1(\mathbb{R}\times\mathbb{R})$, $G(0,0) = 0$ and $G_h(0,0) = 1$.

So by the implicit function theorem, we obtain the existence of a $\varepsilon_0 > 0$ and a function $\bar{h} \in C^1([-\varepsilon_0, \varepsilon_0])$ such that

$$\bar{h}(0) = 0 \qquad \text{and } G\left(\varepsilon, \bar{h}(\varepsilon)\right) = 0 \qquad \text{for every } \varepsilon \in [-\varepsilon_0, \varepsilon_0].$$

Note that the last equation implies

$$\bar{u} + \varepsilon\phi + \bar{h}(\varepsilon)\psi \in X \qquad \text{for every } \varepsilon \in [-\varepsilon_0, \varepsilon_0].$$

Proof of the Lagrange Multiplier theorem III

It also implies (by differentiating),

$$G_{\varepsilon}\left(\varepsilon, \bar{h}(\varepsilon)\right) + G_{h}\left(\varepsilon, \bar{h}(\varepsilon)\right)\bar{h}'(\varepsilon) = 0 \qquad \text{for every } \varepsilon \in [-\varepsilon_0, \varepsilon_0].$$

So we deduce

$$\bar{h}'(0) = -G_{\varepsilon}(0,0)\,. \tag{2}$$

Now once again we use the technique we have already seen. Since $\bar{u}$ is a minimizer, the real valued function on $[-\varepsilon_0, \varepsilon_0]$, given by

$$\varepsilon \mapsto F\left(\varepsilon, \bar{h}(\varepsilon)\right)$$

must have a local minima at $\varepsilon = 0$. So we have

$$0 = \frac{d}{d\varepsilon}\left[F\left(\varepsilon, \bar{h}(\varepsilon)\right)\right]\Big|_{\varepsilon=0} = F_{\varepsilon}(0,0) + F_{h}(0,0)\,\bar{h}'(0).$$

So setting $\lambda = F_h(0,0)$ and using (2), we get

$$F_{\varepsilon}(0,0) - \lambda G_{\varepsilon}(0,0)\,.$$

□

Newton's laws for a point particle

Let $m > 0$ be the mass and $x(t) \in \mathbb{R}^3$ be the position of a point particle. Let $U : \mathbb{R}^3 \to \mathbb{R}$ written as

$$U(t,x) = U(x)$$

be a given potential energy function. The variational problem is

$$m = \inf \left\{ I(x) := \int_0^T f\left(t, x(t), \dot{x}(t)\right) \, \mathrm{d}t : x(0) = x_0, x(T) = x_1 \right\},$$

where the form of the Lagrangian density is

$$f(t,x,\xi) = \frac{1}{2} m\xi^2 - U(x) .$$

The EL equation reads

$$\begin{aligned} 0 &= \frac{d}{dt}\left[f_\xi\left(t, x(t), \dot{x}(t)\right)\right] - f_x\left(t, x(t), \dot{x}(t)\right) \\ &= \frac{d}{dt}\left[m\dot{x}(t)\right] + \nabla U\left(x(t)\right) = m\ddot{x}(t) + \nabla U\left(x(t)\right). \end{aligned}$$

i.e. the Newton's law of motion,

$$m\ddot{x}(t) = -\nabla U\left(x(t)\right) := F\left(x(t)\right).$$

Classical Methods
Classical Problem
Euler-Lagrange Equations
Fundamental Lemma of Calculus of Variations
Derivation of EL equations
Constrained Minimization: Lagrange Multipliers
Example: Newton's laws for a point particle
Hamiltonian formulation
First integrals
Symmetry and Noether's theorem
Hamilton-Jacobi theory and Optimal control
Second Variation
Examples
The End

Thank you

Questions?

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