Introduction to the Calculus of Variations: Lecture 19

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Outline

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Definition (Growth condition on f)

Let $1 < p < \infty$. A Carathéodory function

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for a.e. $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

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for a.e. $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ for some $\alpha_1, \alpha_2 \in L^1(\Omega)$ and $\beta \geq 0$

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Let $n \geq 2, N \geq 1$ *be integers,* $\Omega \subset \mathbb{R}^n$ *be open, bounded, smooth and* $1 < p < \infty$.

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From this, it is easy to establish the uniform L^1 bound.

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Now the RHS clearly is in $L^1(\Omega)$ since $\nabla \bar{u}, \nabla \phi \in L^p\left(\Omega; \mathbb{R}^{N\times n}\right)$. Other terms can be estimated in a similar manner. This completes the proof. □

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for a.e. $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$, where $a, b \geq 0$, $a \in L^1(\mathbb{R}^n)$ and $b \in C\left(\mathbb{R}^N \times \mathbb{R}^{N\times n}\right)$. Let

$$I[u] = I[u, \Omega] := \int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x$$

and suppose there exists $u_0 \in W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$ such that $I[u_0, \Omega] < \infty$. If I is sequentially weakly $*$ lower semicontinuous in $W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$, then

$$\frac{1}{|D|}\int_D f(x_0, u_0, \xi_0 + \nabla\phi(y))\, \mathrm{d}y \geq f(x_0, u_0, \xi_0)$$

for every bounded open set $D \subset \mathbb{R}^n$, for a.e. $x_0 \in \Omega$, for every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and for every $\phi \in W_0^{1,\infty}\left(D; \mathbb{R}^N\right)$.

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Let us now show in a simple setting that

$$\text{convexity} \Rightarrow \text{quasiconvexity}.$$

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This proves f is quasiconvex. □

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$$f(\xi) = \det \xi.$$

Then f is rank one convex but not convex.

Indeed, we have

$$\det \begin{pmatrix} \xi_{11} + ta_1 b_1 & \xi_{12} + ta_1 b_2 \\ \xi_{21} + ta_2 b_1 & \xi_{22} + ta_2 b_2 \end{pmatrix} = (\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) + t\,(a_2 b_2 \xi_{11} + a_1 b_1 \xi_{22} - a_2 b_1 \xi_{12} - a_1 b_2 \xi_{21}).$$

This is clearly affine in t. But clearly, for any $\lambda \in (0,1)$,

$$\lambda(1-\lambda) = \det \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} > \lambda \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (1-\lambda) \det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

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Thank you

Questions?