Introduction to the Calculus of Variations: Lecture 18

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Indian Institute of Science

Spring Semester 2021

Outline

Scorza-Dragoni theorem

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The proof is both delicate and technical, using the Egoroff theorem and the Lusin theorem. Since it would be quite difficult to follow on slides, we relegate the proof to the Lecture Notes.

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Direct methods

Dirichlet Integral

Integrands depending only on the gradient

Integrands with x dependence

Integrands with x and u dependence

Weak lower semicontinuity

Existence of minimizer

Euler-Lagrange Equations

The End

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$$S_\varepsilon := \left\{ (u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n} : |u|\,, |\xi| < M_{\varepsilon_j} \right\}.$$

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$$|\Omega \setminus \Omega_\varepsilon| \leq \sum_{j=1}^{\infty} \left|\Omega \setminus \Omega_{\varepsilon_j, s_{\varepsilon_j}}\right| < \sum_{j=1}^{\infty} \varepsilon_j = \varepsilon \left(\sum_{j=1}^{\infty} \frac{1}{2^j}\right) = \varepsilon.$$

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This proves the claim and finishes the proof of the theorem. □

Existence of minimizer: the general case

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Thank you

Questions?