Introduction to the Calculus of Variations: Lecture 17

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Direct methods
Dirichlet Integral
Integrands depending only on the gradient
Weak lower semicontinuity
Existence of minimizer
Integrands with x dependence
Weak lower semicontinuity
Existence of minimizer
Integrands with x and u dependence
Weak lower semicontinuity
Existence of minimizer
The End

Outline

Direct methods

- Dirichlet Integral
- Integrands depending only on the gradient
  - Weak lower semicontinuity
  - Existence of minimizer
- Integrands with x dependence
  - Weak lower semicontinuity
  - Existence of minimizer
- Integrands with x and u dependence
  - Weak lower semicontinuity
  - Existence of minimizer

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$$\liminf_{s\to\infty} I[u_s] \geq I[u].$$

Mazur lemma

We begin by recalling a standard result in functional analysis whose proof can be found in most textbooks.

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$$y_\mu = \sum_{i=1}^{m_\mu} \alpha_\mu^i x_i \qquad \text{and} \qquad \|y_\mu - x\|_X \to 0 \quad \text{as} \quad \mu \to \infty.$$

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Proof of s.w.l.s.c

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This implies that

$$\liminf_{s\to\infty}\int_\Omega g\left(\nabla u_s(x)\right)\,\mathrm{d}x - \int_\Omega g\left(\nabla u(x)\right)\,\mathrm{d}x$$
$$= \liminf_{s\to\infty}\int_\Omega f\left(\nabla u_s(x)\right)\,\mathrm{d}x - \int_\Omega f\left(\nabla u(x)\right)\,\mathrm{d}x.$$

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Now, since f is convex, for every $\mu \geq 1$, we have

$$f\left(\nabla v_\mu\right) = f\left(\sum_{i=s_0}^{m_\mu} \alpha_\mu^i \nabla u_i\right) \leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i f\left(\nabla u_i\right).$$

Thus, for every $\mu \geq 1$,

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$$\int_\Omega f\left(\nabla v_\mu(x)\right)\,\mathrm{d}x \leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i \int_\Omega f\left(\nabla u_i(x)\right)\,\mathrm{d}x \leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i (L+\varepsilon) = L+\varepsilon.$$

So finally, we deduce

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This completes the proof. □

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If $I[u_0] < \infty$,

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If $I[u_0] < \infty$, *then the following problem*

$$\inf\left\{ I[u] : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\} = m$$

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Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ be given. Let $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\}$ be continuous, convex and satisfies

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admits a minimizer. If f is strictly convex, then the minimizer is unique.

Of course, the simplest and also the prototype example for the last theorem is the p-Dirichlet integral.

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$$\mathcal{D}_p\left[u\right] := \frac{1}{p}\int_{\Omega}\left|\nabla u\left(x\right)\right|^{p}\ \mathrm{d}x.$$

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For $p = 2$, this is the Laplace equation as we have already seen.

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For $p = 2$, this is the Laplace equation as we have already seen. This partial differential operator is also **second order** and **elliptic**, but not **uniformly elliptic**

Of course, the simplest and also the prototype example for the last theorem is the p-Dirichlet integral.

$$\mathcal{D}_p[u] := \frac{1}{p}\int_{\Omega} |\nabla u(x)|^p \, \mathrm{d}x.$$

The Euler-Lagrange equation is the p-Laplacian

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

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This does not fall into the category covered by our theorem since here f depends on x and ξ, and not just ξ.

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As we have already seen in the last example, we want to allow merely measurable dependece on x.

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Definition (Carathéodory functions)

Let $\Omega \subset \mathbb{R}^n$ be open

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Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$.

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Roughly, $f = f(x, \xi)$ is a Carathéodory function when it depends measurably on x for every ξ

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Theorem (sequential weak lower semicontinuity with x dependence)

Let $n \geq 2, N \geq 1$ be integers

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Theorem (sequential weak lower semicontinuity with x dependence)

Let $n \geq 2, N \geq 1$ *be integers and* $1 \leq p < \infty$.

Theorem (sequential weak lower semicontinuity with x dependence)

Let $n \geq 2, N \geq 1$ be integers and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth

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Let $n \geq 2, N \geq 1$ *be integers and* $1 \leq p < \infty$. *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded and smooth and let* $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$, $f = f(x, \xi)$

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$$f(x, \xi) \geq \langle a(x), \xi \rangle + b(x)$$

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Let $n \geq 2, N \geq 1$ be integers and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$, $f = f(x, \xi)$ be a Carathéodory function satisfying

$$f(x, \xi) \geq \langle a(x), \xi \rangle + b(x)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{N \times n}$ for some $a \in L^{p'}(\Omega; \mathbb{R}^{N \times n})$ and some $b \in L^1(\Omega)$. Let

$$I[u] := \int_\Omega f(x, \nabla u(x)) \, \mathrm{d}x.$$

Let $\xi \mapsto f(x, \xi)$ be convex for a.e. $x \in \Omega$ and

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Then we have

$$\liminf_{s\to\infty} I[u_s] \geq I[u].$$

The proof is exactly similar to the last weak lower semicontinuity theorem.

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Existence of minimizer with x dependence

Theorem

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Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$

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Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth.

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Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ be given.

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Any ideas?

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Do not be embarrassed if you did not get this lemon. Most of us have been there. :)

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Unfortunately, our hypotheses still leave out important problems.

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Integrands with x and u dependence

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is a more general important example.

Thank you

Questions?

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