Introduction to the Calculus of Variations: Lecture 16

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Outline

Compactness in L^p spaces

Now we proceed to the question of compactness of the Sobolev embeddings. But before stating the result, we first record a criterion for compactness in $L^q(\Omega)$.

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Lack of compactness at the critical exponent norm

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Proof of Rellich-Kondrachov

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Since $W^{1,2}\left(\Omega;\mathbb{R}^N\right)$ is reflexive, the uniform bound implies that there exists $\bar{u}\in W^{1,2}\left(\Omega;\mathbb{R}^N\right)$ such that up to the extraction of a subsequence, which we do not relabel, we have

$$u_s \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,2}\left(\Omega;\mathbb{R}^N\right).$$

sequential weak lower semicontinuity

Now we wish to prove that

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Since

$$u_s \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,2}\left(\Omega; \mathbb{R}^N\right).$$

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This completes the proof. □

Thank you

Questions?