# Introduction to the Calculus of Variations: Lecture 15

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Now we plan to derive a local version of a Poincaré inequality.

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Lemma (Local Poincaré inequality)

For every  $1 \le p < \infty$ ,

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# Lemma (Local Poincaré inequality)

For every  $1 \le p < \infty$ , there exists a constant c > 0,

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Now we plan to derive a local version of a Poincaré inequality.

Lemma (Local Poincaré inequality)

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Now we plan to derive a local version of a Poincaré inequality.

## Lemma (Local Poincaré inequality)

For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

$$\int_{B(x,r)} |u(y) - u(z)|^p dy \le cr^{n+p-1} \int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y - z|^{n-1}} dy, \quad (1)$$

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for every ball  $B(x,r) \subset \mathbb{R}^n$ , every  $z \in B(x,r)$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

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for every ball  $B(x,r) \subset \mathbb{R}^n$ , every  $z \in B(x,r)$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

### Remark

Note that like the Poincaré inequality,

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# Lemma (Local Poincaré inequality)

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for every ball  $B(x,r) \subset \mathbb{R}^n$ , every  $z \in B(x,r)$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

### Remark

Note that like the Poincaré inequality, here also the estimate controls certain integral related to u by integrals related to  $\nabla u$ .

### Proof.

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**Proof.** We can obviously assume  $u \in C^1(\mathbb{R}^n)$ .

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$$u(y) - u(z) = \int_0^1 \frac{d}{dt} u(z + t(y - z)) dt$$

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$$u(y) - u(z) = \int_0^1 \frac{d}{dt} u(z + t(y - z)) dt$$
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Thus, we have,

$$|u(y) - u(z)|^p \le |y - z|^p \int_0^1 |\nabla u(z + t(y - z))|^p dt.$$
 (2)

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**Proof.** We can obviously assume  $u \in C^1(\mathbb{R}^n)$ . For  $y, z \in B(x, r)$ , we have,

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Let k > 0 be a number such that  $B(x, r) \subset B(z, kr)$  for any  $z \in B(x, r)$ .

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Now, for any s > 0,

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$$\int_{B(x,r)\cap\partial B(z,s)}\left|u\left(y\right)-u\left(z\right)\right|^{p} \mathrm{d}\mathcal{H}^{n-1}\left(y\right)$$

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$$\begin{split} & \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ & \leq s^p \int_0^1 \int_{B(x,r)\cap\partial B(z,s)} \left|\nabla u\left(z+t\left(y-z\right)\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \mathrm{d}t. \end{split}$$

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Putting 
$$w = z + t(y - z)$$

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$$\begin{split} & \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ & \leq s^p \int_0^1 \int_{B(x,r)\cap\partial B(z,s)} \left|\nabla u\left(z+t\left(y-z\right)\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \mathrm{d}t. \end{split}$$

Putting w = z + t(y - z) and changing variables, this implies,

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$$\begin{split} & \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^{p} \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ & \leq s^{p} \int_{0}^{1} \int_{B(x,r)\cap\partial B(z,s)} \left|\nabla u\left(z+t\left(y-z\right)\right)\right|^{p} \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \mathrm{d}t. \end{split}$$

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$$\begin{split} & \int_{B(x,r)\cap\partial B(z,s)} |u\left(y\right) - u\left(z\right)|^{p} \ \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ & \leq s^{p} \int_{0}^{1} \int_{B(x,r)\cap\partial B(z,s)} \left|\nabla u\left(z + t\left(y - z\right)\right)\right|^{p} \ \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \mathrm{d}t. \end{split}$$

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$$\begin{split} & \int_{B(x,r)\cap\partial B(z,s)} |u\left(y\right) - u\left(z\right)|^{p} \ \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ & \leq s^{p} \int_{0}^{1} \int_{B(x,r)\cap\partial B(z,s)} \left|\nabla u\left(z + t\left(y - z\right)\right)\right|^{p} \ \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \mathrm{d}t. \end{split}$$

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$$s^{n+\rho-1}\int_{0}^{1}\frac{1}{\left(ts\right)^{n-1}}\int_{B\left(x,r\right)\cap\partial B\left(z,ts\right)}\left|\nabla u\left(w\right)\right|^{\rho}\;\mathrm{d}\mathcal{H}^{n-1}\left(w\right)\mathrm{d}t$$

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$$s^{n+p-1} \int_0^1 \frac{1}{(ts)^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^p d\mathcal{H}^{n-1}(w) dt$$

$$= s^{n+p-1} \int_0^1 \int_{B(x,r) \cap \partial B(z,ts)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} d\mathcal{H}^{n-1}(w) dt$$

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$$s^{n+p-1} \int_{0}^{1} \frac{1}{(ts)^{n-1}} \int_{B(x,r)\cap\partial B(z,ts)} |\nabla u(w)|^{p} d\mathcal{H}^{n-1}(w) dt$$

$$= s^{n+p-1} \int_{0}^{1} \int_{B(x,r)\cap\partial B(z,ts)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} d\mathcal{H}^{n-1}(w) dt$$

$$= s^{n+p-2} \int_{0}^{s} \int_{B(x,r)\cap\partial B(z,\theta)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} d\mathcal{H}^{n-1}(w) d\theta$$

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$$s^{n+p-1} \int_{0}^{1} \frac{1}{(ts)^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^{p} d\mathcal{H}^{n-1}(w) dt$$

$$= s^{n+p-1} \int_{0}^{1} \int_{B(x,r) \cap \partial B(z,ts)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} d\mathcal{H}^{n-1}(w) dt$$

$$= s^{n+p-2} \int_{0}^{s} \int_{B(x,r) \cap \partial B(z,\theta)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} d\mathcal{H}^{n-1}(w) d\theta$$

$$= s^{n+p-2} \int_{B(x,r) \cap B(z,s)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} dw$$

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$$\begin{split} s^{n+p-1} & \int_{0}^{1} \frac{1}{(ts)^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^{p} \, d\mathcal{H}^{n-1}(w) \, dt \\ & = s^{n+p-1} \int_{0}^{1} \int_{B(x,r) \cap \partial B(z,ts)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} \, d\mathcal{H}^{n-1}(w) \, dt \\ & = s^{n+p-2} \int_{0}^{s} \int_{B(x,r) \cap \partial B(z,ts)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} \, d\mathcal{H}^{n-1}(w) \, d\theta \\ & = s^{n+p-2} \int_{B(x,r) \cap B(z,s)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} \, dw \\ & \leq s^{n+p-2} \int_{B(x,r)} \frac{|\nabla u(w)|^{p}}{|w-z|^{n-1}} \, dw. \end{split}$$

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$$\begin{split} \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ &\leq s^{n+p-2} \int_{B(x,r)} \frac{\left|\nabla u\left(w\right)\right|^p}{\left|w-z\right|^{n-1}} \; \mathrm{d}w. \end{split}$$

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Integrating w.r.t s from 0 to kr

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$$\begin{split} \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ &\leq s^{n+p-2} \int_{B(x,r)} \frac{\left|\nabla u\left(w\right)\right|^p}{\left|w-z\right|^{n-1}} \; \mathrm{d}w. \end{split}$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ ,

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$$\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p d\mathcal{H}^{n-1}(y)$$

$$\leq s^{n+p-2} \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} dw.$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

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$$\begin{split} \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^{p} \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ &\leq s^{n+p-2} \int_{B(x,r)} \frac{\left|\nabla u\left(w\right)\right|^{p}}{\left|w-z\right|^{n-1}} \; \mathrm{d}w. \end{split}$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

$$\int_{B(x,r)} |u(y) - u(z)|^p dy \le \int_{B(x,r) \cap B(z,kr)} |u(y) - u(z)|^p dy$$

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$$\int_{B(x,r)\cap\partial B(z,s)} |u(y) - u(z)|^p d\mathcal{H}^{n-1}(y)$$

$$\leq s^{n+p-2} \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} dw.$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

$$\int_{B(x,r)} |u(y) - u(z)|^p \, \mathrm{d}y \le \int_{B(x,r) \cap B(z,kr)} |u(y) - u(z)|^p \, \mathrm{d}y$$

$$= \int_0^{kr} \int_{B(x,r) \cap \partial B(z,s)} |u(y) - u(z)|^p \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}s$$

Poincaré-Sobolev inequalities

$$\begin{split} \int_{B(x,r)\cap\partial B(z,s)} \left|u\left(y\right)-u\left(z\right)\right|^p \; \mathrm{d}\mathcal{H}^{n-1}\left(y\right) \\ &\leq s^{n+p-2} \int_{B(x,r)} \frac{\left|\nabla u\left(w\right)\right|^p}{\left|w-z\right|^{n-1}} \; \mathrm{d}w. \end{split}$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

$$\begin{split} \int_{B(x,r)} |u(y) - u(z)|^p & \,\mathrm{d}y \leq \int_{B(x,r) \cap B(z,kr)} |u(y) - u(z)|^p & \,\mathrm{d}y \\ &= \int_0^{kr} \int_{B(x,r) \cap \partial B(z,s)} |u(y) - u(z)|^p & \,\mathrm{d}\mathcal{H}^{n-1}\left(y\right) \,\mathrm{d}s \\ &\leq \int_0^{kr} s^{n+p-2} \mathrm{d}s \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} & \,\mathrm{d}w \end{split}$$

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Poincaré-Sobolev inequalities

$$\int_{B(x,r)\cap\partial B(z,s)} |u(y) - u(z)|^p d\mathcal{H}^{n-1}(y)$$

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Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

$$\begin{split} \int_{B(x,r)} |u(y) - u(z)|^p & \,\mathrm{d}y \leq \int_{B(x,r) \cap B(z,kr)} |u(y) - u(z)|^p & \,\mathrm{d}y \\ &= \int_0^{kr} \int_{B(x,r) \cap \partial B(z,s)} |u(y) - u(z)|^p & \,\mathrm{d}\mathcal{H}^{n-1}\left(y\right) \,\mathrm{d}s \\ &\leq \int_0^{kr} s^{n+p-2} \mathrm{d}s \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} & \,\mathrm{d}w \\ &\leq cr^{n+p-1} \int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y-z|^{n-1}} & \,\mathrm{d}y. \end{split}$$

Poincaré-Sobolev inequalities

$$\int_{B(x,r)\cap\partial B(z,s)} |u(y) - u(z)|^p d\mathcal{H}^{n-1}(y)$$

$$\leq s^{n+p-2} \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} dw.$$

Integrating w.r.t s from 0 to kr and noticing that  $B(x,r) \subset B(z,kr)$ , we deduce

$$\begin{split} \int_{B(x,r)} |u(y) - u(z)|^p & \,\mathrm{d}y \leq \int_{B(x,r) \cap B(z,kr)} |u(y) - u(z)|^p & \,\mathrm{d}y \\ &= \int_0^{kr} \int_{B(x,r) \cap \partial B(z,s)} |u(y) - u(z)|^p & \,\mathrm{d}\mathcal{H}^{n-1}\left(y\right) \,\mathrm{d}s \\ &\leq \int_0^{kr} s^{n+p-2} \mathrm{d}s \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} & \,\mathrm{d}w \\ &\leq cr^{n+p-1} \int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y-z|^{n-1}} & \,\mathrm{d}y. \end{split}$$

This proves the lemma.

We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

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# Poincaré inequality with mean on balls

We now prove a Poincaré type inequality for  $\mathcal{W}^{1,p}$  functions.

Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ ,

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The End

We now prove a Poincaré type inequality for  $\mathcal{W}^{1,p}$  functions.

Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ , there exists a constant c > 0,

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We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

# Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p

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he End

We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

# Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

Poincaré-Soholey inequalities

We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

# Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

$$\int_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p dy \le cr^p \int_{B(x,r)} \left| \nabla u(y) \right|^p dy, \quad (3)$$

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for every ball  $B\left(x,r\right)\subset\mathbb{R}^{n}$  and every  $u\in W^{1,p}\left(\mathbb{R}^{n}\right)$ .

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for every ball  $B(x,r) \subset \mathbb{R}^n$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

## Remark

Here the integral mean is

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The End

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For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

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for every ball  $B(x,r) \subset \mathbb{R}^n$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

## Remark

Here the integral mean is

$$(u)_{B(x,r)} := \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy$$

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The End

We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

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For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

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for every ball  $B(x,r) \subset \mathbb{R}^n$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

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Here the integral mean is

$$(u)_{B(x,r)} := \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) dy$$

and the notation for averaged integral is defined as

We now prove a Poincaré type inequality for  $W^{1,p}$  functions.

# Theorem (Poincaré inequality with mean on balls)

For every  $1 \le p < \infty$ , there exists a constant c > 0, depending only on n and p such that

$$\int_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p} dy \le cr^{p} \int_{B(x,r)} \left| \nabla u(y) \right|^{p} dy, \quad (3)$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

## Remark

Here the integral mean is

$$(u)_{B(x,r)} := \frac{1}{|B\left(x,r\right)|} \int_{B(x,r)} u\left(y\right) \; \mathrm{d}y$$

and the notation for averaged integral is defined as

$$\int_{B(x,r)} f(y) \, \mathrm{d}y = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \, \mathrm{d}y.$$

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## Proof.

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**Proof.** As usual we can assume  $u \in C^1(\mathbb{R}^n)$ .

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left(u\left(y\right) - u\left(z\right)\right) \, \mathrm{d}z \right|^p \, \mathrm{d}y \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left( u(y) - u(z) \right) \, \, \mathrm{d}z \right|^p \, \mathrm{d}y \\ & \le \oint_{B(x,r)} \oint_{B(x,r)} \left| u(y) - u(z) \right|^p \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left(u\left(y\right) - u\left(z\right)\right) \, \, \mathrm{d}z \right|^p \, \, \mathrm{d}y \\ & \leq \oint_{B(x,r)} \oint_{B(x,r)} \left| u\left(y\right) - u\left(z\right) \right|^p \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

Now, applying Lemma 1 to estimate the RHS,

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left(u\left(y\right) - u\left(z\right)\right) \, \, \mathrm{d}z \right|^{p} \, \, \mathrm{d}y \\ & \leq \oint_{B(x,r)} \oint_{B(x,r)} \left| u\left(y\right) - u\left(z\right) \right|^{p} \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

Now, applying Lemma 1 to estimate the RHS, we deduce

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left( u\left(y\right) - u\left(z\right) \right) \, \, \mathrm{d}z \right|^p \, \, \mathrm{d}y \\ & \le \oint_{B(x,r)} \oint_{B(x,r)} \left| u\left(y\right) - u\left(z\right) \right|^p \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

Now, applying Lemma 1 to estimate the RHS, we deduce

$$\int_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p} dy$$

$$\leq c \int_{B(x,r)} r^{p-1} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^{p}}{\left| y - z \right|^{n-1}} dz dy$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left( u\left(y\right) - u\left(z\right) \right) \, \, \mathrm{d}z \right|^p \, \, \mathrm{d}y \\ & \le \oint_{B(x,r)} \oint_{B(x,r)} \left| u\left(y\right) - u\left(z\right) \right|^p \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & \leq c \oint_{B(x,r)} r^{p-1} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^p}{\left| y - z \right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^p}{\left| y - z \right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & = \oint_{B(x,r)} \left| \oint_{B(x,r)} \left( u\left(y\right) - u\left(z\right) \right) \, \, \mathrm{d}z \right|^p \, \, \mathrm{d}y \\ & \le \oint_{B(x,r)} \oint_{B(x,r)} \left| u\left(y\right) - u\left(z\right) \right|^p \, \, \mathrm{d}y \mathrm{d}z \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\ & \leq c \oint_{B(x,r)} r^{p-1} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^p}{\left| y - z \right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^p}{\left| y - z \right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left|\nabla u\left(z\right)\right|^{p}}{\left|y - z\right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left|\nabla u\left(z\right)\right|^{p}}{\left|y - z\right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & = c r^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \oint_{B(x,r)} \frac{1}{\left|y - z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left| \nabla u(z) \right|^{p}}{\left| y - z \right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & = c r^{p-1} \int_{B(x,r)} \left| \nabla u(z) \right|^{p} \left( \oint_{B(x,r)} \frac{1}{\left| y - z \right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & \leq c r^{p-1} \int_{B(x,r)} \left| \nabla u(z) \right|^{p} \left( \frac{1}{r^{n}} \int_{B(z,kr)} \frac{1}{\left| y - z \right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \end{split}$$

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$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq cr^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left|\nabla u\left(z\right)\right|^{p}}{\left|y - z\right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & = cr^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \oint_{B(x,r)} \frac{1}{\left|y - z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & \leq cr^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \frac{1}{r^{n}} \int_{B(z,kr)} \frac{1}{\left|y - z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & = c \frac{r^{p}}{r^{n}} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \mathrm{d}z \end{split}$$

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### Now using Fubini, we deduce

$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq c r^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left|\nabla u\left(z\right)\right|^{p}}{\left|y - z\right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & = c r^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \oint_{B(x,r)} \frac{1}{\left|y - z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & \leq c r^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \frac{1}{r^{n}} \int_{B(z,kr)} \frac{1}{\left|y - z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & = c \frac{r^{p}}{r^{n}} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \, \mathrm{d}z \\ & = c r^{p} \oint_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \, \mathrm{d}z. \end{split}$$

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Now using Fubini, we deduce

$$\begin{split} & \oint_{B(x,r)} \left| u\left(y\right) - \left(u\right)_{B(x,r)} \right|^{p} \, \mathrm{d}y \\ & \leq cr^{p-1} \oint_{B(x,r)} \int_{B(x,r)} \frac{\left|\nabla u\left(z\right)\right|^{p}}{\left|y-z\right|^{n-1}} \, \mathrm{d}z \mathrm{d}y \\ & = cr^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \oint_{B(x,r)} \frac{1}{\left|y-z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & \leq cr^{p-1} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \left( \frac{1}{r^{n}} \int_{B(z,kr)} \frac{1}{\left|y-z\right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\ & = c\frac{r^{p}}{r^{n}} \int_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \mathrm{d}z \\ & = cr^{p} \oint_{B(x,r)} \left|\nabla u\left(z\right)\right|^{p} \, \mathrm{d}z. \end{split}$$

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# Poincaré-Sobolev inequality with mean on balls

mean on balls.

As a corollary, we derive the Poincaré-Sobolev inequality with

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every  $1 \le p < n$ ,

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every  $1 \le p < n$ , there exists a constant c > 0,

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every  $1 \le p < n$ , there exists a constant c > 0, depending only on n and p such that

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Poincaré-Sobolev inequality with mean on balls

As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

# Theorem (Poincaré-Sobolev inequality with mean on balls)

For every  $1 \le p < n$ , there exists a constant c > 0, depending only on n and p such that

$$\left(\int_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p^*} dy \right)^{\frac{1}{p^*}} \le cr \left(\int_{B(x,r)} \left| \nabla u(y) \right|^{p} dy \right)^{\frac{1}{p}}, \tag{4}$$

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

# Theorem (Poincaré-Sobolev inequality with mean on balls)

For every  $1 \le p < n$ , there exists a constant c > 0, depending only on n and p such that

$$\left(\int_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p^*} dy \right)^{\frac{1}{p^*}} \le cr \left(\int_{B(x,r)} \left| \nabla u(y) \right|^{p} dy \right)^{\frac{1}{p}}, \tag{4}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and every  $u \in W^{1,p}(\mathbb{R}^n)$ .

### Proof.

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$$\left( \int_{B(x,r)} |v(y)|^{p^*} dy \right)^{\frac{1}{p^*}}$$

$$\leq c \left( r^p \int_{B(x,r)} |\nabla v(y)|^p dy + \int_{B(x,r)} |v(y)|^p dy \right)^{\frac{1}{p}},$$

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$$\begin{split} \left( f_{B(x,r)} \left| v\left(y\right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v\left(y\right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v\left(y\right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

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$$\begin{split} \left( f_{B(x,r)} \left| v\left(y\right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v\left(y\right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v\left(y\right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$ 

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$$\begin{split} \left( f_{B(x,r)} \left| v\left(y\right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v\left(y\right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v\left(y\right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$  and translation,

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$$\begin{split} \left( f_{B(x,r)} \left| v\left(y\right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v\left(y\right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v\left(y\right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$  and translation, we can assume that x=0 and r=1.

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$$\begin{split} \left( \int_{B(x,r)} \left| v \left( y \right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v \left( y \right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v \left( y \right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$  and translation, we can assume that x=0 and r=1. But in this case, the inequality above is just the Poincaré-Sobolev inequality for the bounded domain

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$$\begin{split} \left( f_{B(x,r)} \left| v\left(y\right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v\left(y\right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v\left(y\right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$  and translation, we can assume that x=0 and r=1. But in this case, the inequality above is just the Poincaré-Sobolev inequality for the bounded domain  $B(0,1) \subset \mathbb{R}^n$ .

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$$\begin{split} \left( f_{B(x,r)} \left| v \left( y \right) \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} \left| \nabla v \left( y \right) \right|^p \, \mathrm{d}y + \int_{B(x,r)} \left| v \left( y \right) \right|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \end{split}$$

for every ball  $B(x,r) \subset \mathbb{R}^n$  and for every  $v \in W^{1,p}(\mathbb{R}^n)$  with  $1 \leq p < n$ .

Note that replacing v by  $\frac{1}{r}v(ry)$  and translation, we can assume that x=0 and r=1. But in this case, the inequality above is just the Poincaré-Sobolev inequality for the bounded domain  $B(0,1) \subset \mathbb{R}^n$ .

This proves the inequality.

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Now we apply this inequality to the function  $v := u - (u)_{B(x,r)}$ .

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Now we apply this inequality to the function  $v := u - (u)_{B(x,r)}$ . We obtain

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Now we apply this inequality to the function  $v:=u-(u)_{\mathcal{B}(\mathsf{x},r)}$  . We obtain

$$\left( \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}}$$

$$\leq c \left( r^p \int_{B(x,r)} \left| \nabla u \right|^p + \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^p \right)^{\frac{1}{p}}.$$

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Now we apply this inequality to the function  $v:=u-(u)_{B(x,r)}$  . We obtain

$$\left( \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}} \\
\leq c \left( r^p \int_{B(x,r)} |\nabla u|^p + \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^p \right)^{\frac{1}{p}}.$$

Now we use the Poincaré inequality with mean on balls

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Now we apply this inequality to the function  $v:=u-(u)_{\mathcal{B}(\mathsf{x},r)}$  . We obtain

$$\begin{split} \left( \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}} \\ & \leq c \left( r^p \int_{B(x,r)} |\nabla u|^p + \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^p \right)^{\frac{1}{p}}. \end{split}$$

Now we use the Poincaré inequality with mean on balls to estimate the last term to obtain

$$\left(\int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left( r^p \int_{B(x,r)} \left| \nabla u \right|^p \right)^{\frac{1}{p}}.$$

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Now we apply this inequality to the function  $v:=u-(u)_{B(\mathsf{x},r)}$  . We obtain

$$\left( \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}} \\
\leq c \left( r^p \int_{B(x,r)} |\nabla u|^p + \int_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^p \right)^{\frac{1}{p}}.$$

Now we use the Poincaré inequality with mean on balls to estimate the last term to obtain

$$\left(\int_{B(x,r)}\left|u-(u)_{B(x,r)}\right|^{p^*}\right)^{\frac{1}{p^*}}\leq c\left(r^p\int_{B(x,r)}\left|\nabla u\right|^p\right)^{\frac{1}{p}}.$$

This proves the theorem.

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# Morrey's inequality

Now we prove an important inequality.

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Now we prove an important inequality.

Theorem (Morrey's inequality)

For every n ,

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Now we prove an important inequality.

Theorem (Morrey's inequality)

For every n , there exists a constant <math>c > 0,

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Now we prove an important inequality.

Theorem (Morrey's inequality)

For every n , there exists a constant <math>c > 0, depending only on n and p such that

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Now we prove an important inequality.

## Theorem (Morrey's inequality)

For every n , there exists a constant <math>c > 0, depending only on n and p such that

$$|u(y) - u(z)| \le cr \left( \int_{B(x,r)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{5}$$

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Now we prove an important inequality.

# Theorem (Morrey's inequality)

For every n , there exists a constant <math>c > 0, depending only on n and p such that

$$|u(y) - u(z)| \le cr \left( \int_{B(x,r)} |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{5}$$

for a.e.  $y, z \in B(x, r)$  for every ball  $B(x, r) \subset \mathbb{R}^n$  and for every  $u \in W^{1,p}(\mathbb{R}^n)$ .

We use the local Poincaré inequality lemma with p=1

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We use the local Poincaré inequality lemma with p=1 to deduce

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Rellich-Kondrachov compact embeddings We use the local Poincaré inequality lemma with p=1 to deduce

$$|u(y)-u(z)|$$

$$\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw$$

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We use the local Poincaré inequality lemma with p=1 to deduce

$$|u(y)-u(z)|$$

$$\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw 
\leq c \int_{B(x,r)} |\nabla u(w)| (|y - w|^{1-n} + |z - w|^{1-n}) dw$$

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 $\overset{\mathsf{H\"{o}lder}}{\leq} c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{captardo} \, \mathsf{Nrewberg}}{\mathsf{captardo} \, \mathsf{Nrewberg}}} \right)^{\frac{p}{p-1}} \, \mathrm{d} w \right)^{\frac{p}{p-1}} \, \mathrm{d} w$ 

$$\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw 
\leq c \int_{B(x,r)} |\nabla u(w)| (|y - w|^{1-n} + |z - w|^{1-n}) dw$$

|u(y) - u(z)|

 $\stackrel{\text{H\"older}}{\leq} c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{2p-1}{p}} \, \mathrm{d}v \right)^{\frac{p-1}{p}} dv$ 

 $\leq cr^{1-\frac{n}{p}}\left(\int_{B(x,r)}|\nabla u(w)|^p dw\right)^{\frac{1}{p}}.$ 

 $\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw$ 

 $\leq c\int_{B(x,r)}\left|\nabla u\left(w\right)\right|\left(\left|y-w\right|^{1-n}+\left|z-w\right|^{1-n}\right)\;\mathrm{d}w$ 

|u(y) - u(z)|

|u(y) - u(z)|

$$\leq \int_{B(x,r)} \left( |u(y) - u(w)| + |u(w) - u(z)| \right) dw$$

$$\leq c \int_{B(x,r)} \left| \nabla u(w) \right| \left( \left| y - w \right|^{1-n} + \left| z - w \right|^{1-n} \right) dw$$

Hölder 
$$\leq c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u \begin{pmatrix} w \end{pmatrix}^{1-p} \, \mathrm{d}w \end{pmatrix}^{\frac{p-1}{p}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \, \mathrm{d}w$$

$$\leq c r^{1-\frac{n}{p}} \left( \int_{B(x,r)} |\nabla u (w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} \, .$$
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This proves the inequality.

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 $\stackrel{\text{H\"{o}lder}}{\leq} c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{2p-1}{p}} \, \mathrm{d}w \right)^{\frac{2p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{2p-1}{p}} \, \mathrm{d}w \right)^{\frac{2p-1}{p}} \, \mathrm{d}w$ 

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$$|u(y)-u(z)|$$

$$\leq \int_{B(x,r)} \left( |u(y) - u(w)| + |u(w) - u(z)| \right) dw$$

$$\leq c \int_{B(x,r)} \left| \nabla u(w) \right| \left( \left| y - w \right|^{1-n} + \left| z - w \right|^{1-n} \right) dw$$

$$\leq cr^{1-\frac{n}{p}}\left(\int_{B(x,r)}|\nabla u(w)|^p dw\right)^{\frac{1}{p}}.$$

This proves the inequality.

Note that in the last line above,

 $\overset{\mathsf{H\"{o}lder}}{\leq} c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p-1}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w \right)^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}} \left( \int_{B(x,r)} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} |\nabla u|^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}} \right)^{\frac{\mathsf{constrainty}}{\mathsf{constrainty}}$ 

|u(y) - u(z)|

$$\leq \int_{B(x,r)} \left( \left| u(y) - u(w) \right| + \left| u(w) - u(z) \right| \right) dw$$

$$\leq c \int_{B(x,r)} \left| \nabla u(w) \right| \left( \left| y - w \right|^{1-n} + \left| z - w \right|^{1-n} \right) dw$$

$$\leq cr^{1-\frac{n}{p}}\left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w\right)^{\frac{1}{p}}.$$

This proves the inequality.

Note that in the last line above, we used the convexity of the function  $t \mapsto t^{\frac{p}{p-1}}$ 

|u(y) - u(z)|

$$\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw$$

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$$\overset{\mathsf{H\"{o}lder}}{\leq} c \left( \int_{\mathcal{B}(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{\mathcal{B}(x,r)} |\nabla u \left( \underbrace{w} \right)|^{\frac{2}{p} + \frac{2}{p} + \frac{2}{p}}_{\mathsf{d}} \right)^{\frac{2}{p} + \frac{2}{p}} \left( \int_{\mathcal{B}(x,r)} |\nabla u \left( w \right)|^{p} \, \mathrm{d}w \right)^{\frac{1}{p}} \, \mathrm{d}w$$

This proves the inequality.

Note that in the last line above, we used the convexity of the function  $t \mapsto t^{\frac{p}{p-1}}$  and the estimate

Calculus of Variations

$$|u(y)-u(z)|$$

$$\leq \int_{B(x,r)} \left( \left| u\left(y\right) - u\left(w\right) \right| + \left| u\left(w\right) - u\left(z\right) \right| \right) \, \mathrm{d}w$$

$$\leq c \int_{B(x,r)} \left| \nabla u\left(w\right) \right| \left( \left| y - w \right|^{1-n} + \left| z - w \right|^{1-n} \right) \, \mathrm{d}w$$

$$\overset{\text{H\"{o}lder}}{\leq} c \left( \int_{\mathcal{B}(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{\mathcal{B}(x,r)} |\nabla u \begin{pmatrix} u \\ u \end{pmatrix}^{1-n} \, \mathrm{d}v \end{pmatrix}^{\frac{1}{p}}$$

$$\overset{\text{Lagrano-virenberg-Sobolev inequalities}}{\leq c r^{1-\frac{n}{p}}} \left( \int_{\mathcal{B}(x,r)} |\nabla u (w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} .$$

$$\overset{\text{Lagrano-virenberg-Sobolev inequalities}}{\leq c r^{1-\frac{n}{p}}} \left( \int_{\mathcal{B}(x,r)} |\nabla u (w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} .$$

$$\overset{\text{The End}}{\leq c r^{1-\frac{n}{p}}} \left( \int_{\mathcal{B}(x,r)} |\nabla u (w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} .$$

This proves the inequality.

Note that in the last line above, we used the convexity of the function  $t \mapsto t^{\frac{p}{p-1}}$  and the estimate

$$\int_{B(x,r)} |y - w|^{\frac{\rho(1-n)}{\rho-1}} \, \mathrm{d} w \le \int_{B(y,kr)} |y - w|^{\frac{\rho(1-n)}{\rho-1}} \, \mathrm{d} w$$

 $\stackrel{\text{H\"{o}lder}}{\leq} c \left( \int_{B(x,r)} \left( |y-w|^{1-n} + |z-w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left( \int_{B(x,r)} |\nabla u \begin{pmatrix} v \\ w \end{pmatrix} \right)^{\frac{p-1}{p-1}} \, \mathrm{d}w$ 

We use the local Poincaré inequality lemma with p=1 to deduce

$$|u(y) - u(z)|$$

$$\leq \int_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) dw$$

$$\leq c \int_{B(x,r)} |\nabla u(w)| \left( |y-w|^{1-n} + |z-w|^{1-n} \right) dw$$

$$\leq cr^{1-\frac{n}{p}}\left(\int_{B(x,r)}\left|\nabla u\left(w\right)\right|^{p} \mathrm{d}w\right)^{\frac{1}{p}}.$$

This proves the inequality.

Note that in the last line above, we used the convexity of the function  $t \mapsto t^{\frac{p}{p-1}}$  and the estimate

$$\int_{B(x,r)} |y - w|^{\frac{\rho(1-n)}{\rho-1}} dw \le \int_{B(y,kr)} |y - w|^{\frac{\rho(1-n)}{\rho-1}} dw$$

$$= \int_{0}^{kr} \int_{\mathbb{S}^{n-1}} \rho^{(n-1)\left(1 - \frac{\rho}{\rho-1}\right)} d\rho d\theta = cr^{\frac{\rho-n}{\rho-1}}.$$

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Hölder continuous with exponent  $\alpha = 1 - \frac{n}{n}$ .

Morrey's inequality implies that  $W^{1,p}$  functions with p > n are

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Morrey's inequality implies that  $W^{1,p}$  functions with p>n are Hölder continuous with exponent  $\alpha=1-\frac{n}{p}$ .

Theorem (Sobolev embedding in  $\mathbb{R}^n$  for p > n)

Let n .

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Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

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# Theorem (Sobolev embedding in $\mathbb{R}^n$ for p > n)

Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

## Proof.

By Morrey's inequality,

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Morrey's inequality implies that  $W^{1,p}$  functions with p > n are Hölder continuous with exponent  $\alpha = 1 - \frac{n}{p}$ .

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Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

#### Proof.

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# Theorem (Sobolev embedding in $\mathbb{R}^n$ for p > n)

Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

#### Proof.

$$|u(x) - u(y)| \le cr^{1-\frac{n}{\rho}} \left( \int_{B(x,2r)} |\nabla u(w)|^{\rho} dw \right)^{\frac{1}{\rho}}$$

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Morrey's inequality implies that  $W^{1,p}$  functions with p > n are Hölder continuous with exponent  $\alpha = 1 - \frac{n}{n}$ .

# Theorem (Sobolev embedding in $\mathbb{R}^n$ for p > n)

Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

#### Proof.

$$|u(x) - u(y)| \le cr^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$
$$= c|x - y|^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$

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Morrey's inequality implies that  $W^{1,p}$  functions with p > n are Hölder continuous with exponent  $\alpha = 1 - \frac{n}{n}$ .

# Theorem (Sobolev embedding in $\mathbb{R}^n$ for p > n)

Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

## Proof.

$$|u(x) - u(y)| \le cr^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$

$$= c |x - y|^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$

$$\le c |x - y|^{1-\frac{n}{p}} \left\| \nabla u \right\|_{L^p(\mathbb{R}^n)}.$$

Morrey's inequality implies that  $W^{1,p}$  functions with p>n are Hölder continuous with exponent  $\alpha=1-\frac{n}{p}$ .

# Theorem (Sobolev embedding in $\mathbb{R}^n$ for p > n)

Let  $n . Then <math>W^{1,p}(\mathbb{R}^n)$  continuously embeds into  $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ .

## Proof.

By Morrey's inequality, for a.e.  $x,y\in\mathbb{R}^n$  with |x-y|=r, we have,

$$|u(x) - u(y)| \le cr^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$

$$= c |x - y|^{1-\frac{n}{p}} \left( \int_{B(x,2r)} |\nabla u(w)|^p \, dw \right)^{\frac{1}{p}}$$

$$\le c |x - y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Hölder continuity follows ( See Lecture notes for details ).

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Theorem (Sobolev embedding in bounded domains for p > n)

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n .

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Theorem (Sobolev embedding in bounded domains for p > n)

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n . $Then <math>W^{1,p}(\Omega)$  continuously embeds into  $C^{0,\alpha}(\overline{\Omega})$ 

Let 
$$\Omega \subset \mathbb{R}^n$$
 be open, bounded and smooth and let  $n .Then  $W^{1,p}\left(\Omega\right)$  continuously embeds into  $C^{0,\alpha}\left(\overline{\Omega}\right)$  for every  $0 \le \alpha \le 1 - \frac{n}{p}$ .$ 

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As a consequence, we can deduce

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**Theorem** (
$$W^{1,\infty} = C^{0,1}$$
)

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth.

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n .

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth. Then

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 (with equivalent norms).

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## Proof.

Since  $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ 

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**Theorem** (
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Since  $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$  for any n , by the last theorem,

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n . $Then <math>W^{1,p}(\Omega)$  continuously embeds into  $C^{0,\alpha}(\overline{\Omega})$  for every

As a consequence, we can deduce

Theorem (
$$W^{1,\infty} = C^{0,1}$$
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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth. Then

$$W^{1,\infty}\left(\Omega\right)=C^{0,1}\left(\overline{\Omega}\right)$$
 (with equivalent norms).

## Proof.

Since  $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$  for any  $n , by the last theorem, for any <math>x, y \in \overline{\Omega}$ , we obtain

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n . $Then <math>W^{1,p}(\Omega)$  continuously embeds into  $C^{0,\alpha}(\overline{\Omega})$  for every

 $0 \le \alpha \le 1 - \frac{n}{p}$ .

As a consequence, we can deduce

Theorem ( $W^{1,\infty} = C^{0,1}$ )

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth. Then

$$W^{1,\infty}\left(\Omega\right)=C^{0,1}\left(\overline{\Omega}\right)$$
 (with equivalent norms).

## Proof.

Since  $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$  for any  $n , by the last theorem, for any <math>x, y \in \overline{\Omega}$ , we obtain

$$|u(x) - u(y)| \le c |x - y|^{1 - \frac{n}{p}} ||u||_{W^{1,p}(\Omega)}.$$

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Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth and let n . $Then <math>W^{1,p}(\Omega)$  continuously embeds into  $C^{0,\alpha}(\overline{\Omega})$  for every

$$0 \le \alpha \le 1 - \frac{n}{p}$$
.

As a consequence, we can deduce

Theorem (
$$W^{1,\infty} = C^{0,1}$$
)

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and smooth. Then

$$W^{1,\infty}\left(\Omega\right)=C^{0,1}\left(\overline{\Omega}\right)$$
 (with equivalent norms).

## Proof.

Since  $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$  for any  $n , by the last theorem, for any <math>x,y \in \overline{\Omega}$ , we obtain

$$|u(x) - u(y)| \le c |x - y|^{1 - \frac{n}{p}} ||u||_{W^{1,p}(\Omega)}.$$

Letting  $p \to \infty$ ,

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Rellich-Kondrachov compact embeddings

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Letting  $p \to \infty$ , we obtain  $W^{1,\infty}\left(\Omega\right) \subset C^{0,1}\left(\overline{\Omega}\right)$ . The other inclusion is easy and was proved earlier in this chapter.

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# Thank you Questions?

# Introduction to the Calculus of Variations

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