Introduction to the Calculus of Variations: Lecture 15

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Spring Semester 2021

Sobolev spaces

Definitions

Elementary properties

Approximation and extension

Traces

Sobolev inequalities and Sobolev embeddings

Gagliardo-Nirenberg-Sobolev inequalities

Poincaré-Sobolev inequalities

Morrey's inequality

Rellich-Kondrachov compact embeddings

The End

Outline

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- Definitions
- Elementary properties
- Approximation and extension
- Traces
- Sobolev inequalities and Sobolev embeddings
  - Gagliardo-Nirenberg-Sobolev inequalities
  - Poincaré-Sobolev inequalities
  - Morrey's inequality
  - Rellich-Kondrachov compact embeddings

Poincaré inequality on balls

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For every $1 \leq p < \infty$, *there exists a constant* $c > 0$,

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Now we plan to derive a local version of a Poincaré inequality.

Lemma (Local Poincaré inequality)

For every $1 \leq p < \infty$, there exists a constant $c > 0$, depending only on n and p such that

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$$\int_{B(x,r)} |u(y) - u(z)|^p \, \mathrm{d}y \leq c r^{n+p-1} \int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y - z|^{n-1}} \, \mathrm{d}y, \quad (1)$$

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for every ball $B\left(x,r\right) \subset \mathbb{R}^{n}$, every $z \in B\left(x,r\right)$ and every $u \in W^{1,p}\left(\mathbb{R}^{n}\right)$.

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Remark

Note that like the Poincaré inequality,

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Remark

Note that like the Poincaré inequality, here also the estimate controls certain integral related to u by integrals related to ∇u.

Proof.

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Now, for any $s > 0$,

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The RHS of the last inequality is

$$s^{n+p-1}\int_0^1 \frac{1}{(ts)^{n-1}}\int_{B(x,r)\cap\partial B(z,ts)} |\nabla u(w)|^p \, \mathrm{d}\mathcal{H}^{n-1}(w)\,\mathrm{d}t$$

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$$\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p \,\mathrm{d}\mathcal{H}^{n-1}(y) \leq s^{n+p-2}\int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}}\,\mathrm{d}w.$$

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Integrating w.r.t s from 0 to kr

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Integrating w.r.t s from 0 to kr and noticing that $B(x,r) \subset B(z,kr)$, we deduce

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$$\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p \, \mathrm{d}\mathcal{H}^{n-1}(y) \leq s^{n+p-2}\int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} \, \mathrm{d}w.$$

Integrating w.r.t s from 0 to kr and noticing that $B(x,r) \subset B(z,kr)$, we deduce

$$\int_{B(x,r)} |u(y)-u(z)|^p \, \mathrm{d}y \leq \int_{B(x,r)\cap B(z,kr)} |u(y)-u(z)|^p \, \mathrm{d}y$$

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So we arrive at

$$\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p \, \mathrm{d}\mathcal{H}^{n-1}(y) \leq s^{n+p-2}\int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}}\,\mathrm{d}w.$$

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This proves the lemma. □

Poincaré inequality with mean on balls

We now prove a Poincaré type inequality for $W^{1,p}$ functions.

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Theorem (Poincaré inequality with mean on balls)

For every $1 \leq p < \infty$,

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$$(u)_{B(x,r)} := \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \, \mathrm{d}y$$

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$$\fint_{B(x,r)} f(y) \, \mathrm{d}y = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \, \mathrm{d}y.$$

Proof.

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Now, applying Lemma 1 to estimate the RHS,

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$$\begin{aligned}
&\fint_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right|^p \, \mathrm{d}y \\
&\quad \leq c \fint_{B(x,r)} r^{p-1} \int_{B(x,r)} \frac{\left|\nabla u(z)\right|^p}{\left|y - z\right|^{n-1}} \, \mathrm{d}z\mathrm{d}y
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&\quad = c r^{p-1} \int_{B(x,r)} \left| \nabla u(z) \right|^p \left(\fint_{B(x,r)} \frac{1}{\left| y - z \right|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z
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Now using Fubini, we deduce

$$\begin{aligned}
&\fint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p} \mathrm{d}y \\
&\leq cr^{p-1} \fint_{B(x,r)} \int_{B(x,r)} \frac{|\nabla u(z)|^{p}}{|y-z|^{n-1}} \,\mathrm{d}z\mathrm{d}y \\
&= cr^{p-1} \int_{B(x,r)} |\nabla u(z)|^{p} \left(\fint_{B(x,r)} \frac{1}{|y-z|^{n-1}} \,\mathrm{d}y \right) \mathrm{d}z \\
&\leq cr^{p-1} \int_{B(x,r)} |\nabla u(z)|^{p} \left(\frac{1}{r^{n}} \int_{B(z,kr)} \frac{1}{|y-z|^{n-1}} \,\mathrm{d}y \right) \mathrm{d}z
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&= cr^{p-1} \int_{B(x,r)} |\nabla u(z)|^p \left(\fint_{B(x,r)} \frac{1}{|y-z|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\
&\leq cr^{p-1} \int_{B(x,r)} |\nabla u(z)|^p \left(\frac{1}{r^n} \int_{B(z,kr)} \frac{1}{|y-z|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\
&= c\frac{r^p}{r^n} \int_{B(x,r)} |\nabla u(z)|^p \, \mathrm{d}z
\end{aligned}$$

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&= cr^{p-1} \int_{B(x,r)} |\nabla u(z)|^{p} \left(\fint_{B(x,r)} \frac{1}{|y-z|^{n-1}} \,\mathrm{d}y \right) \mathrm{d}z \\
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&= c\frac{r^{p}}{r^{n}} \int_{B(x,r)} |\nabla u(z)|^{p} \,\mathrm{d}z \\
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&= cr^p \fint_{B(x,r)} |\nabla u(z)|^p \,\mathrm{d}z.
\end{aligned}$$

□

Poincaré-Sobolev inequality with mean on balls

As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

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Poincaré-Sobolev inequality with mean on balls

As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every $1 \leq p < n$,

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every $1 \leq p < n$, there exists a constant $c > 0$,

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As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every $1 \leq p < n$, there exists a constant $c > 0$, depending only on n and p such that

Poincaré-Sobolev inequality with mean on balls

As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem (Poincaré-Sobolev inequality with mean on balls)

For every $1 \leq p < n$, there exists a constant $c > 0$, depending only on n and p such that

$$\left(\fint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \leq c r \left(\fint_{B(x,r)} \left| \nabla u(y) \right|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}}, \tag{4}$$

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for every ball $B(x,r) \subset \mathbb{R}^n$ and every $u \in W^{1,p}(\mathbb{R}^n)$.

Proof.

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$$\left(\fint_{B(x,r)} |v(y)|^{p^*} \, \mathrm{d}y \right)^{\frac{1}{p^*}} \leq c \left(r^p \fint_{B(x,r)} |\nabla v(y)|^p \, \mathrm{d}y + \fint_{B(x,r)} |v(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}},$$

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Note that replacing v by $\frac{1}{r} v(ry)$ and translation, we can assume that $x = 0$ and $r = 1$. But in this case, the inequality above is just the Poincaré-Sobolev inequality for the bounded domain $B(0,1) \subset \mathbb{R}^n$.

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This proves the inequality.

Now we apply this inequality to the function $v := u - (u)_{B(x,r)}$.

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Now we use the Poincaré inequality with mean on balls to estimate the last term to obtain

$$\left(\fint_{B(x,r)} \left| u - (u)_{B(x,r)} \right|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left(r^p \fint_{B(x,r)} |\nabla u|^p \right)^{\frac{1}{p}} .$$

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This proves the theorem. □

Morrey's inequality

Now we prove an important inequality.

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Now we prove an important inequality.

Theorem (Morrey's inequality)

For every $n < p < \infty,$

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Theorem (Morrey's inequality)

For every $n < p < \infty$, *there exists a constant* $c > 0$,

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Theorem (Morrey's inequality)

For every $n < p < \infty$, there exists a constant $c > 0$, depending only on n and p such that

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Now we prove an important inequality.

Theorem (Morrey's inequality)

For every $n < p < \infty$, there exists a constant $c > 0$, depending only on n and p such that

$$|u(y) - u(z)| \leq cr \left(\fint_{B(x,r)} |\nabla u(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \qquad (5)$$

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for a.e. $y, z \in B(x,r)$ for every ball $B(x,r) \subset \mathbb{R}^n$ and for every $u \in W^{1,p}(\mathbb{R}^n)$.

We use the local Poincaré inequality lemma with $p = 1$

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We use the local Poincaré inequality lemma with $p = 1$ to deduce

$$|u(y) - u(z)|$$

$$\leq \fint_{B(x,r)} \left(|u(y) - u(w)| + |u(w) - u(z)| \right) \, \mathrm{d}w$$

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&|u(y) - u(z)| \\
&\leq \fint_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) \,\mathrm{d}w \\
&\leq c \int_{B(x,r)} |\nabla u(w)| \left(|y - w|^{1-n} + |z - w|^{1-n} \right) \,\mathrm{d}w
\end{aligned}
$$

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&\overset{\text{Hölder}}{\leq} c \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n}\right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^{p} \, \mathrm{d}w \right)^{\frac{1}{p}}
\end{aligned}$$

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&\overset{\text{Hölder}}{\leq} c \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n}\right)^{\frac{p}{p-1}} \,\mathrm{d}w\right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \,\mathrm{d}w\right)^{\frac{1}{p}} \\
&\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \,\mathrm{d}w\right)^{\frac{1}{p}}.
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We use the local Poincaré inequality lemma with $p=1$ to deduce

$$
\begin{aligned}
&\left|u(y)-u(z)\right| \\
&\leq \fint_{B(x,r)} \left(\left|u(y)-u(w)\right| + \left|u(w)-u(z)\right|\right)\, \mathrm{d}w \\
&\leq c\int_{B(x,r)} \left|\nabla u(w)\right| \left(\left|y-w\right|^{1-n} + \left|z-w\right|^{1-n}\right)\, \mathrm{d}w \\
&\overset{\text{Hölder}}{\leq} c\left(\int_{B(x,r)} \left(\left|y-w\right|^{1-n} + \left|z-w\right|^{1-n}\right)^{\frac{p}{p-1}}\, \mathrm{d}w\right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} \left|\nabla u(w)\right|^{p}\, \mathrm{d}w\right)^{\frac{1}{p}} \\
&\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,r)} \left|\nabla u(w)\right|^{p}\, \mathrm{d}w\right)^{\frac{1}{p}}.
\end{aligned}
$$

This proves the inequality. □

We use the local Poincaré inequality lemma with $p = 1$ to deduce

$$|u(y) - u(z)|$$

$$\leq \fint_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|)\, \mathrm{d}w$$

$$\leq c \int_{B(x,r)} |\nabla u(w)| \left(|y - w|^{1-n} + |z - w|^{1-n} \right) \mathrm{d}w$$

$$\overset{\text{Hölder}}{\leq} c \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \mathrm{d}w \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}$$

$$\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}.$$

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Note that in the last line above,

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&\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p\, \mathrm{d}w\right)^{\frac{1}{p}}.
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Note that in the last line above, we used the convexity of the function $t \mapsto t^{\frac{p}{p-1}}$

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&\leq \fint_{B(x,r)} \left(\left|u\left(y\right)-u\left(w\right)\right|+\left|u\left(w\right)-u\left(z\right)\right|\right)\,\mathrm{d}w \\
&\leq c\int_{B(x,r)} \left|\nabla u\left(w\right)\right|\left(\left|y-w\right|^{1-n}+\left|z-w\right|^{1-n}\right)\,\mathrm{d}w \\
&\overset{\text{Hölder}}{\leq} c\left(\int_{B(x,r)}\left(\left|y-w\right|^{1-n}+\left|z-w\right|^{1-n}\right)^{\frac{p}{p-1}}\,\mathrm{d}w\right)^{\frac{p-1}{p}}\left(\int_{B(x,r)}\left|\nabla u\left(w\right)\right|^{p}\,\mathrm{d}w\right)^{\frac{1}{p}} \\
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$$\int_{B(x,r)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w \leq \int_{B(y,kr)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w$$

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$$|u(y) - u(z)|$$
$$\leq \fint_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) \, \mathrm{d}w$$
$$\leq c \int_{B(x,r)} |\nabla u(w)| \left(|y - w|^{1-n} + |z - w|^{1-n} \right) \, \mathrm{d}w$$
$$\stackrel{\text{Hölder}}{\leq} c \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}$$
$$\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}.$$

This proves the inequality. □

Note that in the last line above, we used the convexity of the function $t \mapsto t^{\frac{p}{p-1}}$ and the estimate

$$\int_{B(x,r)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w \leq \int_{B(y,kr)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w$$
$$= \int_0^{kr} \int_{\mathbb{S}^{n-1}} \rho^{(n-1)\left(1-\frac{p}{p-1}\right)} \, \mathrm{d}\rho \, \mathrm{d}\theta = c r^{\frac{p-n}{p-1}}.$$

Sobolev embedding for $p > n$

Morrey's inequality implies that $W^{1,p}$ functions with $p > n$ are Hölder continuous with exponent $\alpha = 1 - \frac{n}{p}$.

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Theorem (Sobolev embedding in $\mathbb{R}^n$ for $p > n$)

Let $n < p < \infty$.

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Let $n < p < \infty$. Then $W^{1,p}(\mathbb{R}^n)$ continuously embeds into $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

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Proof.

By Morrey's inequality,

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Let $n < p < \infty$. Then $W^{1,p}(\mathbb{R}^n)$ continuously embeds into $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.

Proof.

By Morrey's inequality, for a.e. $x, y \in \mathbb{R}^n$ with $|x - y| = r$, we have,

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$$|u(x) - u(y)| \leq cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}$$

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Hölder continuity follows (See Lecture notes for details). □

Theorem (Sobolev embedding in bounded domains for $p > n$)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth

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Theorem (Sobolev embedding in bounded domains for $p > n$)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $n < p < \infty$.

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Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $n < p < \infty$. Then $W^{1,p}(\Omega)$ continuously embeds into $C^{0,\alpha}\left(\overline{\Omega}\right)$

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Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $n < p < \infty$. Then $W^{1,p}(\Omega)$ continuously embeds into $C^{0,\alpha}(\overline{\Omega})$ for every $0 \leq \alpha \leq 1 - \frac{n}{p}$.

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Proof.

Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$

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Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

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Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

$$|u(x) - u(y)| \leq c\,|x - y|^{1-\frac{n}{p}} \|u\|_{W^{1,p}(\Omega)}.$$

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Proof.

Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

$$|u(x) - u(y)| \leq c\,|x-y|^{1-\frac{n}{p}}\,\|u\|_{W^{1,p}(\Omega)}\,.$$

Letting $p \to \infty$,

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Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

$$|u(x) - u(y)| \leq c\,|x - y|^{1-\frac{n}{p}} \|u\|_{W^{1,p}(\Omega)}.$$

Letting $p \to \infty$, we obtain $W^{1,\infty}(\Omega) \subset C^{0,1}(\overline{\Omega})$.

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Proof.

Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

$$|u(x) - u(y)| \leq c\,|x - y|^{1 - \frac{n}{p}} \|u\|_{W^{1,p}(\Omega)}.$$

Letting $p \to \infty$, we obtain $W^{1,\infty}(\Omega) \subset C^{0,1}(\overline{\Omega})$. The other inclusion is easy and was proved earlier in this chapter. □

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Thank you

Questions?