Introduction to the Calculus of Variations: Lecture 14

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Outline

Theorem (Gagliardo-Nirenberg-Sobolev inequality)

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To prove this inequality, we need a simple lemma.

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$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

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Integrating w.r.t x_{n+1} and Hölder inequality gives the result. □

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This proves the theorem. □

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Thank you

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