Introduction to the Calculus of Variations: Lecture 13

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Spring Semester 2021

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In this section we are going to study two results.

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- ► The approximation of a Sobolev function in a bounded smooth domain, in the Sobolev norm, by functions which are smooth up to the boundary.

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Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary.

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Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Then for any $1 \leq p \leq \infty$, there exists a linear extension operator

$$P:W^{1,p}\left(\Omega\right)\to W^{1,p}\left(\mathbb{R}^{n}\right)$$

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where the constant c > 0 depends only on Ω .

Global approximation by smooth functions

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Theorem (Global approximation by smooth functions)

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$$\left\{ u_{s}\right\} _{s=1}^{\infty}\subset W^{1,p}\left(\Omega\right) \cap C^{\infty}\left(\overline{\Omega}\right)$$

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Remark

The result is false for $p = \infty$.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Let $u \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence

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 in $W^{1,p}(\Omega)$.

Remark

The result is false for $p = \infty$.

Clearly, this result follows from the extension result by mollification.

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$$\Phi: \overline{B_1(0)} \to \overline{U_{x_0}}$$

satisfying

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This basically is the coordinate change that maps the point x_0 to the origin, maps the portion of Ω in U_{x_0} to the upper half ball B_1^+ (0), maps the portion of $\partial\Omega$ in U_{x_0} to the portion of the equatorial hyperplane in the unit ball and takes the inward normal to $\partial\Omega$ to the postive direction of the x_n coordinate.

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Thus, if we care only about a small neighborhood of a boundary point, we can transfer our problem to extending from the upper half-ball to the whole ball and then transfer back.

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Now the question is, can we somehow 'cut' u into pieces near the boundary, work with each piece separately and then finally patch them up?

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for some integer M>0 and some neighborhoods U_{x_i} of the boundary points $x_1,\ldots,x_M\in\partial\Omega$.

To 'cut' *u* into pieces,

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Proposition (partition of unity)

Let Γ be a compact subset of \mathbb{R}^n

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Proposition (partition of unity)

Let Γ be a compact subset of \mathbb{R}^n and let U_1, \ldots, U_M be a finite open covering of Γ .

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 on \mathbb{R}^n ,

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Moreover, if $\Omega \subset \mathbb{R}^n$ is an open bounded set such that $\Gamma = \partial \Omega$,

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$$\zeta_0|_{\Omega} \in C_c^{\infty}(\Omega)$$
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How do we use this to 'cut' *u* into pieces?

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 for $0 \le i \le M$,

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Patching up the pieces

On the other hand, if we are given functions $v_i \in W^{1,p}\left(U_i\right)$

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Localizing or cutting into pieces

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 for $0 \le i \le M$,

are the pieces of u.

Patching up the pieces

On the other hand, if we are given functions $v_i \in W^{1,p}(U_i)$ for every $0 \le i \le M$, then

$$v:=\sum_{i=0}^{M}\zeta_{i}v_{i}\in W^{1,p}\left(\mathbb{R}^{n}\right).$$

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- cut u into pieces as discussed,
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- cut u into pieces as discussed,
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- cut u into pieces as discussed,
- locally flatten the boundary, i.e. compose each piece of u near the boundary with the respective diffemorphisms to obtain Sobolev functions defined on the upper half ball,
- extend those Sobolev functions to the whole ball,

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- cut u into pieces as discussed,
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- ▶ patch all these pieces together to obtain a Sobolev function on the whole of \mathbb{R}^n .

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Note that the u_0 piece lives in the interior of Ω ,

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To carry out the plan,

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To carry out the plan, all that remains is to figure out how to extend a Sobolev function on the upper half ball

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To carry out the plan, all that remains is to figure out how to extend a Sobolev function on the upper half ball **which vanishes near the curved part of the boundary**

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Note that the u_0 piece lives in the interior of Ω , so we can just extend it by zero.

To carry out the plan, all that remains is to figure out how to extend a Sobolev function on the upper half ball **which vanishes near the curved part of the boundary** from the upper half ball to the whole ball.

Extension by reflection

At the level of $W^{1,p}$, it is hardly surprising or difficult.

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At the level of $W^{1,p}$, it is hardly surprising or difficult. We just use reflection across the flat part of the boundary of the upper half ball.

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Lemma Let $u \in W^{1,p}(B_1^+(0))$,

At the level of $W^{1,p}$, it is hardly surprising or difficult. We just use reflection across the flat part of the boundary of the upper half

Lemma

ball.

Let $u \in W^{1,p}\left(B_1^+\left(0\right)\right)$, where $1 \leq p \leq \infty$ be such that it vanishes near the curved part of the boundary,

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Lemma

Let $u \in W^{1,p}\left(B_1^+\left(0\right)\right)$, where $1 \le p \le \infty$ be such that it vanishes near the curved part of the boundary, i.e. $\partial B_1\left(0\right) \cap \{x_n > 0\}$.

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$$\tilde{u}(x',x_n) := \begin{cases} u(x',x_n) & \text{if } x_n > 0, \\ u(x',-x_n) & \text{if } x_n < 0. \end{cases}$$

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belongs to $W^{1,p}(B_1(0))$, extends u to $B_1(0)$ and vanishes near $\partial B_1(0)$.

At the level of $W^{1,p}$, it is hardly surprising or difficult. We just use reflection across the flat part of the boundary of the upper half ball.

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The value of the function obviously matches and perhaps slightly less obviously, the tangential derivatives along the equatorial hyperplane match too. So the only thing to check is whether the normal derivative matches across the equatorial hyperplane $\{x_n = 0\}$. You are asked to prove this in the problem sheets.

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Now we want to tackle the problem of defining 'boundary values'

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Theorem (Existence of Trace operator)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary

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Theorem (Existence of Trace operator)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary and let $1 \leq p < \infty$.

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As before, by using localization, flattening the boundary and patching up, the proof of the theorem can be reduced to proving the following.

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We define the operator for smooth functions by assigning the boundary values. This operator is clearly linear. Since $C^{\infty}\left(\overline{\Omega}\right)$ is dense in $W^{1,p}\left(\Omega\right)$, to show boundedness, it is enough to show the estimate

$$\left\|u\right|_{\partial\Omega}\right\|_{L^{p}(\partial\Omega)}\leq c\left\|u\right\|_{W^{1,p}(\Omega)}\quad \text{ for every } u\in C^{\infty}\left(\overline{\Omega}\right).$$

As before, by using localization, flattening the boundary and patching up, the proof of the theorem can be reduced to proving the following.

Lemma

There exists a constant c > 0 such that

$$\|u|_{\partial\Omega}\|_{L^{p}(\partial\Omega)} \le c \|u\|_{W^{1,p}(\Omega)}$$
 for every $u \in C^{\infty}\left(\overline{\Omega}\right)$.

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Lemma

$$\left(\int_{\mathbb{R}^{n-1}}\left|u\left(x',0\right)\right|^{p} \mathrm{~d}x'\right)^{\frac{1}{p}} \leq c \left\|u\right\|_{W^{1,p}\left(\mathbb{R}^{n}_{+}\right)} \quad \text{ for every } u \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right).$$

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Proof.

Let
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, for all $t \in \mathbb{R}$.

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Taking absolute values and then Young's inequality, this implies

$$|u(x',0)|^{p} \leq p \int_{0}^{+\infty} |u(x',x_{n})|^{p-1} \left| \frac{\partial u}{\partial x_{n}} (x',x_{n}) \right| dx_{n}$$

$$\leq c \left(\int_{0}^{+\infty} |u(x',x_{n})|^{p} dx_{n} + \int_{0}^{+\infty} \left| \frac{\partial u}{\partial x_{n}} (x',x_{n}) \right|^{p} dx_{n} \right).$$

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 $\leq c \left(\int_{\hat{x}}^{+\infty} |u(x',x_n)|^p dx_n + \int_{\hat{x}}^{+\infty} \left| \frac{\partial u}{\partial x_n} (x',x_n) \right|^p dx_n \right).$

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pause = $-\int_{-\infty}^{+\infty} F'(u(x',x_n)) \frac{\partial u}{\partial x}(x',x_n) dx_n$

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The lemma follows by integrating w.r.t. $x' \in \mathbb{R}^{n-1}$ and taking p-th roots

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 $F(u(x',0)) = -\int_{0}^{+\infty} \frac{\partial}{\partial x_{n}} F(u(x',x_{n})) dx_{n}$

 $|u(x',0)|^p \le p \int_0^{+\infty} |u(x',x_n)|^{p-1} \left| \frac{\partial u}{\partial x_n}(x',x_n) \right| dx_n$

p-th roots along with obvious estimates.

 $F(t) := |t|^{p-1} t$, for all $t \in \mathbb{R}$.

 $\leq c\left(\int_{\hat{x}}^{+\infty}\left|u\left(x',x_{n}\right)\right|^{p} \mathrm{d}x_{n}+\int_{\hat{x}}^{+\infty}\left|\frac{\partial u}{\partial x_{n}}\left(x',x_{n}\right)\right|^{p} \mathrm{d}x_{n}\right).$

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$$\operatorname{Ker}(T_0) = W_0^{1,p}(\Omega).$$

Figuring out the exact image of the trace map is delicate.

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Higher order traces can be defined similarly and requires more Sobolev regularity for those traces to be in $L^p(\partial\Omega)$.

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We begin our discussion with the case $1 \le p < n$.

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Suppose we want to prove an inequality of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \le c \|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \text{ for all } u \in C_c^\infty(\mathbb{R}^n).$$
 (4)

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What can the exponent q be?

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So, (4) applied to u_{λ} implies

$$||u||_{L^{q}(\mathbb{R}^{n})} \leq c\lambda^{\left(1-\frac{n}{p}+\frac{n}{q}\right)} ||\nabla u||_{L^{p}(\mathbb{R}^{n})}.$$
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Remark

Note that we always have $p^* > p$.

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Theorem (Gagliardo-Nirenberg-Sobolev inequality) Let $1 \le p \le n$.

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Theorem (Gagliardo-Nirenberg-Sobolev inequality)

Let $1 \le p < n$. Then there exists a constant c > 0,

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$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\right)^{\frac{1}{p^*}} \le c \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}} \tag{6}$$

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We shall prove this inequality in the next lecture.

Thank you Questions?

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