Introduction to the Calculus of Variations: Lecture 12

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Outline

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Thus, we calculate

$$\begin{aligned} 0 &= \frac{d}{dt}E_{\dot{\gamma}_i} - E_{\gamma_i} \\ &= \frac{d}{dt}\left[2g_{ij}(\gamma(t))\,\dot{\gamma}^j(t)\right] - \left(\frac{\partial}{\partial z^i}g_{kj}\right)(\gamma(t))\,\dot{\gamma}^k(t)\,\dot{\gamma}^j(t) \\ &= 2g_{ij}\ddot{\gamma}^j + 2\frac{\partial}{\partial z^k}g_{ij}\dot{\gamma}^k\dot{\gamma}^j - \frac{\partial}{\partial z^i}g_{kj}\dot{\gamma}^k\dot{\gamma}^j. \end{aligned}$$

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We say that u is *weakly differentiable* if the weak partial derivatives $u_{x_1}, \cdots, u_{x_n}$ exist.

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This would imply that $\varphi(0) = 0$, for every $\varphi \in C_0^\infty(-1,1)$, which is clearly absurd. Thus H is not weakly differentiable.

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Examples

Now we present some examples which are instructive.

Example

Let $s > 0$,

$$\Omega = \{x \in \mathbb{R}^n : |x| < 1\} \quad \text{and} \quad \psi(x) = |x|^{-s}.$$

We then have

$$\psi \in L^p \Leftrightarrow sp < n \quad \text{and} \quad \psi \in W^{1,p} \Leftrightarrow (s+1)p < n.$$

This is a typical one about how singular a Sobolev function can be. Another similar one is the following.

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then $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ for every $1 \leq p < n$.

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We are not going to prove the Meyer-Serrin theorem. The proof of the density result with regular enough boundary would be sketched later.

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Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p \leq \infty$ and $u \in L^p(\Omega)$. The following properties are then equivalent.

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$$\left| \int_a^b u\,\varphi' \right| \leq \gamma \left\| \varphi \right\|_{L^{p'}(a,b)}\,, \quad \forall\, \varphi \in C_0^\infty\left(a,b\right)$$

which is exactly (ii).

Note that the γ appearing in (iii) and in (ii) can be taken the same.

(i) $\Rightarrow$ **(iii)**. We have for every $x \in \omega$

$$u\left(x+\tau\right) - u\left(x\right) = \int_x^{x+\tau} u'\left(t\right) dt = \tau \int_0^1 u'\left(x+s\tau\right) ds$$

and hence

$$\left|u\left(x+\tau\right) - u\left(x\right)\right| \leq |\tau| \int_0^1 \left|u'\left(x+s\tau\right)\right| ds.$$

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$$|u\,(x+\tau) - u\,(x)| \leq |\tau| \int_0^1 |u'\,(x+s\tau)|\,ds.$$

Let $1 < p < \infty$ (the conclusion is obvious if $p = \infty$), we have from Jensen inequality that

$$|u\,(x+\tau) - u\,(x)|^p \leq |\tau|^p \int_0^1 |u'\,(x+s\tau)|^p\,ds$$

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$$\begin{aligned}\int_{\omega} \left|u\left(x+\tau\right)-u\left(x\right)\right|^{p} dx &\leq \left|\tau\right|^{p} \int_{\omega}\int_{0}^{1} \left|u'\left(x+s\tau\right)\right|^{p} ds\, dx \\ &= \left|\tau\right|^{p} \int_{0}^{1}\int_{\omega} \left|u'\left(x+s\tau\right)\right|^{p} dx\, ds.\end{aligned}$$

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Since $\omega + s\tau \subset \left(a, b\right)$, we find

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Since $\omega + s\tau \subset (a, b)$, we find

$$\int_{\omega} \left| u'\left(x+s\tau\right) \right|^{p} dx = \int_{\omega+s\tau} \left| u'\left(y\right) \right|^{p} dy \leq \left\| u' \right\|_{L^{p}(a,b)}^{p}$$

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Hence after integration

$$\int_{\omega} |u(x+\tau) - u(x)|^p \, dx \leq |\tau|^p \int_{\omega} \int_0^1 |u'(x+s\tau)|^p \, ds \, dx$$
$$= |\tau|^p \int_0^1 \int_{\omega} |u'(x+s\tau)|^p \, dx \, ds.$$

Since $\omega + s\tau \subset (a,b)$, we find

$$\int_{\omega} |u'(x+s\tau)|^p \, dx = \int_{\omega+s\tau} |u'(y)|^p \, dy \leq \|u'\|_{L^p(a,b)}^p$$

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Hence after integration

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Since $\omega + s\tau \subset (a,b)$, we find

$$\int_{\omega} |u'(x+s\tau)|^p \, dx = \int_{\omega+s\tau} |u'(y)|^p \, dy \leq \|u'\|^p_{L^p(a,b)}$$

and hence

$$\|D_\tau u\|_{L^p(\omega)} \leq \|u'\|_{L^p(a,b)}$$

which is the claim.

Thank you

Questions?