Introduction to the Calculus of Variations: Lecture 11

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Prelude to Direct Methods
Geodesics: the problem
Absolute continuity: first encounter with Sobolev spaces
Existence of geodesics
Regularity questions
The End

Outline

Prelude to Direct Methods

- Geodesics: the problem
- Absolute continuity: first encounter with Sobolev spaces
- Existence of geodesics
- Regularity questions

Poincaré inequality in $W_0^{1,p}$

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whenever $(\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)$ are disjoint segments in (a, b).

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We shall prove it in the problem sheet in stages.

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Indeed,

$$\begin{aligned} L(c \circ \tau) &= \int_0^S \left| \frac{d}{ds} (c \circ \tau)(s) \right| \, \mathrm{d}s \\ &= \int_0^S \left| \left(\frac{d}{dt} c \right) (\tau(s)) \right| \left| \frac{d\tau}{ds} (s) \right| \, \mathrm{d}s \\ &= \int_0^T \left| \dot{c}(t) \right| \, \mathrm{d}t = L(c) . \end{aligned}$$

Reparametrization and group action

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Just for an analogy, suppose we are looking to find the unit interval $[0, 1]$. Suppose, however, that our problem is invariant under the **action of $\mathbb{Z}$**, i.e. invariant under the transformation

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Probably it is better to view the analogy in reverse.

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Remark

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- *Any injective, rectifiable, absolutely continuous curve can be (re)parametrized by arc-length.*

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Thus, we can minimize $E(c)$ instead of $L(c)$ to find geodesic curves.

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Now we settle the problem of the existence of a geodesic.

Theorem (existence of geodesics)

Assume $f : U \subset \mathbb{R}^N \to M \subset \mathbb{R}^d$ be a chart on M such that the metric tensor is uniformly positive definite in $f(U)$ and let $p_1 \neq p_2 \in M$ be contained in the image of f. Suppose there exists at least one Lipschitz curve c_0 on $f(U)$ joining p_1 and p_2 and let

$$D := \inf \left\{ L(c) : c \text{ is a Lipschitz curve on } f(U) \text{ joining } p_1 \text{ and } p_2 \right\}.$$

Then the variational problem

$$\inf_{\gamma \in \gamma_0 + W_0^{1,2}([0,D];U)} \left\{ I(\gamma) := \frac{1}{2} \int_0^D g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) \, \mathrm{d}t \right\} = \frac{1}{2} D,$$

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Note that here we have used the fact that $W_0^{1,2}$, being a convex subset of $W^{1,2}$, is weakly closed.

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Note that here we have used the fact that $W_0^{1,2}$, being a convex subset of $W^{1,2}$, is weakly closed. However, here in dimension one, we could have also used the fact that

$$\gamma_\nu \rightharpoonup \gamma \quad \text{in } W^{1,2} \qquad \Rightarrow \qquad \gamma_\nu \to \gamma \quad \text{in } C^0.$$

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Now we want to show that this weak limit γ is a minimizer.

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Hence γ is a minimizer. □

Regularity

Now we are going to show that this curve is actually C^2 and not just $W^{1,2}$.

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Then u is C^2. Moreover, if f_ξ is C^k for some $k \geq 2$, then u is C^{k+1}. In particular, u is C^∞ if f_ξ is C^∞.

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Proof. We define the function $\phi : [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as

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Thank you

Questions?

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