Introduction to the Calculus of Variations: Lecture 10

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Outline

Recap

We have already defined weak derivatives.

Definition (weak derivatives)

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Any two weak derivatives of u would be equal a.e. by the fundamental lemma of calculus of variations and thus would represent the same L^1 function.

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The completion of $X^{1,p}$ with respect to the above norm is called $H^{1,p}\left((a,b);\mathbb{R}^N\right)$.

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Clearly, $U = U_1 + U_2$ does the job.

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This is something we have already seen implicitly in attempting to solve the geodesic problem before.

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Theorem (Characterization of $W_0^{1,p}$)

Let $u \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$. Then $u \in W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$ if and only if $u(a) = 0 = u(b)$.

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$$u_\nu = \frac{1}{\nu} G(\nu u),$$

so that $u_\nu \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$.

Proof

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$$u_\nu \to u \qquad \text{in } W^{1,p}\left((a,b);\mathbb{R}^N\right)$$

by the dominated convergence theorem. □

Thank you

Questions?