Introduction to the Calculus of Variations: Lecture 1

Swarnendu Sil

Department of Mathematics
Indian Institute of Science

Spring Semester 2021

Welcome to
" Introduction to the Calculus of Variations"

Hello everyone
A warm welcome to you all

Hello everyone
A warm welcome to you all

I am Swarnendu.
I work on Calculus of Variations, PDEs and Geometric Analysis.
I obtained my PhD in 2016 from EPFL, Switzerland.

Hello everyone
A warm welcome to you all

I am Swarnendu.
I work on Calculus of Variations, PDEs and Geometric Analysis.
I obtained my PhD in 2016 from EPFL, Switzerland.

This course is a gentle introduction to
the direct methods in Calculus of Variations concerning minimization problems.

I intend to give you a flavour of the subject using important prototype examples.
So we will mostly not care about proving the sharpest result.

Practical information

- Course Page Link: http://math.iisc.ac.in/~ssil/Courses/Intro_CalcVar_Spring21/CalcVar1Spring21.html. All the material related to the course would be on this page.
- A detailed course content outline and prerequisites can be found at http://math.iisc.ac.in/~ssil/Courses/Intro_CalcVar_Spring21/Intro_Calc_Var_2021_Course_Syllabus.pdf
- Lectures would take place on Microsoft Teams. Slides and lecture notes will be uploaded to the course page as soon as possible once the lecture is over, ideally within a few minutes.

A few words about the course

- The course has three integral parts,
  - **Lecture and/or Slides**,
  - **Lecture Notes** and
  - **Problem Sheets** (along with **Solution keys** which would be posted later, ideally the week after).

  Each of these are absolutely indispensable. They are not copies of one another, though significant overlaps are of course unavoidable. There are lots of material in the lecture notes which might not even be mentioned in the slides. There are many problems in the assignments the likes which are not discussed in neither the slides nor the lecture notes.
- Draw lots of figures and sketches! Our pattern-friendly brains love those incomparably more than greek alphabets.
- Treat any phrase like ‘clearly’, ‘it is easy to see’ etc as a bull would treat a red flag. Do not let me slide a shaky argument past you! Faith has no place in mathematics and the only authority it recognizes is that of an iron-clad proof.

Outline of the Course

Skeleton

The course is divided into six chapters as follows.

- **Chapter 1: Introduction**
- **Chapter 2: Classical Methods**
- **Chapter 3: Tools from Analysis**
- **Chapter 4: Direct Methods**
- **Chapter 5: Regularity**
- **Chapter 6: Plateau's problem and Minimal surfaces**

Outline of the Course

Skeleton

The course is divided into six chapters as follows.

- **Chapter 1: Introduction**
- **Chapter 2: Classical Methods**
- **Chapter 3: Tools from Analysis**
- **Chapter 4: Direct Methods**
- **Chapter 5: Regularity**
- **Chapter 6: Plateau's problem and Minimal surfaces**

The first chapter would be quite short. The second chapter discusses the classical methods. The third chapter basically collects some analytic preliminaries for later chapters. The last three chapters are in some sense the heart of the course.

Chapter 1: Introduction

Basic case of finding a minima

We begin with the study of a considerably simpler problem, which we all learned how to solve in our Calculus courses - finding a minima of a given function. However, this would already give us a glimpse of what is to come.

We begin with the study of a considerably simpler problem, which we all learned how to solve in our Calculus courses - finding a minima of a given function. However, this would already give us a glimpse of what is to come.

Finding the minima of a function

Let $X \subset \mathbb{R}^n$ and let $F : X \to \mathbb{R}$. Consider the minimization problem

$$\inf\{F(x) : x \in X\} \tag{P}$$

Basic case of finding a minima

We begin with the study of a considerably simpler problem, which we all learned how to solve in our Calculus courses - finding a minima of a given function. However, this would already give us a glimpse of what is to come.

Finding the minima of a function

Let $X \subset \mathbb{R}^n$ and let $F : X \to \mathbb{R}$. Consider the minimization problem

$$\inf\{F(x) : x \in X\} \tag{P}$$

There are roughly two methods to solve the problem (if it can be solved at all!!)

Basic case of finding a minima

We begin with the study of a considerably simpler problem, which we all learned how to solve in our Calculus courses - finding a minima of a given function. However, this would already give us a glimpse of what is to come.

Finding the minima of a function

Let $X \subset \mathbb{R}^n$ and let $F : X \to \mathbb{R}$. Consider the minimization problem

$$\inf\{F(x) : x \in X\} \tag{P}$$

There are roughly two methods to solve the problem (if it can be solved at all!!)
The methods are called **the classical method** and **the direct method**.

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1.

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

Try to solve this to obtain all critical points.

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

Try to solve this to obtain all critical points. Then check for a local minima.

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

Try to solve this to obtain all critical points. Then check for a local minima.
Now if F is C^2,

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \tag{1}$$

Try to solve this to obtain all critical points. Then check for a local minima.
Now if F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. positive definite.} \quad \textbf{(Legendre condition)} \tag{2}$$

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \tag{1}$$

Try to solve this to obtain all critical points. Then check for a local minima.
Now if F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. positive definite.} \quad \textbf{(Legendre condition)} \tag{2}$$

This is exactly how we found minima in Calculus courses!

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \tag{1}$$

Try to solve this to obtain all critical points. Then check for a local minima.
Now if F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. positive definite.} \quad \textbf{(Legendre condition)} \tag{2}$$

This is exactly how we found minima in Calculus courses!
Note that if F is strictly convex and C^2, (2) is automatic.

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

Try to solve this to obtain all critical points. Then check for a local minima.
Now if F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. positive definite.} \quad \textbf{(Legendre condition)} \qquad (2)$$

This is exactly how we found minima in Calculus courses!
Note that if F is strictly convex and C^2, (2) is automatic.
The minima would also be unique and global.

Basic case of finding a minima: The classical method

- **Classical Method** Assume F is C^1. Then $\bar{x} \in X$ is a stationary point/critical point of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \qquad \textbf{(Euler-Lagrange Eqn.)} \qquad (1)$$

  Try to solve this to obtain all critical points. Then check for a local minima.
  Now if F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. positive definite.} \quad \textbf{(Legendre condition)} \qquad (2)$$

  This is exactly how we found minima in Calculus courses!
  Note that if F is strictly convex and C^2, (2) is automatic. The minima would also be unique and global.

Convexity in general plays an important role in minization problems.
See Lecture notes and the assignments for more on this.

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s \to \infty} F(x_s) = \inf\{F(x) : x \in X\}. \tag{3}$$

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s \to \infty} F(x_s) = \inf\{F(x) : x \in X\}. \tag{3}$$

Such sequences must exist, by definition of the infimum!

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s\to\infty} F(x_s) = \inf\{F(x) : x \in X\}. \qquad (3)$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences.

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s\to\infty} F(x_s) = \inf\{F(x) : x \in X\}. \qquad (3)$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences. Typically, we can deduce an uniform bound

$$\|x_s\| \leq C \qquad \text{for all } s.$$

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s\to\infty} F(x_s) = \inf\{F(x) : x \in X\}. \qquad (3)$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences. Typically, we can deduce an uniform bound

$$\|x_s\| \leq C \qquad \text{for all } s.$$

Then up to the extraction of a subsequence that we do not relabel, we get

$$x_s \to \bar{x}, \qquad \text{for some } \bar{x} \in X. \quad \text{(by Bolzano-Weierstrass)} \qquad (4)$$

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s\to\infty} F(x_s) = \inf\{F(x) : x \in X\}. \tag{3}$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences. Typically, we can deduce an uniform bound

$$\|x_s\| \leq C \qquad \text{for all } s.$$

Then up to the extraction of a subsequence that we do not relabel, we get

$$x_s \to \bar{x}, \qquad \text{for some } \bar{x} \in X. \quad \text{(by Bolzano-Weierstrass)} \tag{4}$$

Now if F is lower semicontinuous, we obtain

$$\inf\{F(x) : x \in X\} \overset{\text{since } \bar{x}\in X}{\leq} F(\bar{x}) \overset{\text{lsc}}{\leq} \lim_{s\to\infty} F(x_s) \overset{\text{min seq}}{=} \inf\{F(x) : x \in X\}.$$

Basic case of finding a minima: The direct method

- **Direct Method** Let $\{x_s\} \subset X$ be a **minimizing sequence**, i.e.

$$\lim_{s\to\infty} F(x_s) = \inf\{F(x) : x \in X\}. \tag{3}$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences. Typically, we can deduce an uniform bound

$$\|x_s\| \leq C \qquad \text{for all } s.$$

Then up to the extraction of a subsequence that we do not relabel, we get

$$x_s \to \bar{x}, \qquad \text{for some } \bar{x} \in X. \quad \text{(by Bolzano-Weierstrass)} \tag{4}$$

Now if F is lower semicontinuous, we obtain

$$\begin{aligned}\inf\{F(x) : x \in X\} &\overset{\text{since } \bar{x}\in X}{\leq} F(\bar{x}) \\ &\overset{\text{lsc}}{\leq} \lim_{s\to\infty} F(x_s) \overset{\text{min seq}}{=} \inf\{F(x) : x \in X\}.\end{aligned}$$

Thus all inequalities must be equalities and $\bar{x}$ is a minima.

Basic case of finding a minima: The direct method

Probably you have never used this method before to find a minima!

Basic case of finding a minima: The direct method

Probably you have never used this method before to find a minima! So it is tempting to conclude that the direct method is somehow inferior. No!

Basic case of finding a minima: The direct method

Probably you have never used this method before to find a minima! So it is tempting to conclude that the direct method is somehow inferior. No!

Consider $f(x) = |x|$. A brief glance at its graph is enough to see that it has a unique, global minima at the origin.

Basic case of finding a minima: The direct method

Probably you have never used this method before to find a minima! So it is tempting to conclude that the direct method is somehow inferior. No!
Consider $f(x) = |x|$. A brief glance at its graph is enough to see that it has a unique, global minima at the origin.

Direct method needs only **lower semicontinuity**. There are lots of lower semicontinuous functions! Many of them are discontinuous!

Basic case of finding a minima: The direct method

Probably you have never used this method before to find a minima! So it is tempting to conclude that the direct method is somehow inferior. No!
Consider $f(x) = |x|$. A brief glance at its graph is enough to see that it has a unique, global minima at the origin.

Direct method needs only **lower semicontinuity**. There are lots of lower semicontinuous functions! Many of them are discontinuous!

Obtaining the uniform bound for minimizing sequences is not always easy. But these generally follow from **coercivity** assumptions on F. For example, $F : \mathbb{R}^n \to \mathbb{R}$ is said to have **superlinear growth at infinity** if it satisfies

$$\lim_{\|x\| \to \infty} \frac{|F(x)|}{\|x\|} = +\infty.$$

You are asked to show in the assignment that this implies a uniform bound for minimizing sequences if F is bounded below.

Basic case of finding a minima: Comparison of the methods

Comparing the two methods

- The classical method

Comparing the two methods

- The classical method
  - works by solving the EL equations

Basic case of finding a minima: Comparison of the methods

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points

Basic case of finding a minima: Comparison of the methods

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular

Basic case of finding a minima: Comparison of the methods

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular
- The direct method

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular
- The direct method
  - works directly with the function, not the EL equations (Hence the name!)

Basic case of finding a minima: Comparison of the methods

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular
- The direct method
  - works directly with the function, not the EL equations (Hence the name!)
  - finds only minima, not all critical points (There are direct methods for finding other critical points too, but those are beyond the scope of this course)

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular
- The direct method
  - works directly with the function, not the EL equations (Hence the name!)
  - finds only minima, not all critical points (There are direct methods for finding other critical points too, but those are beyond the scope of this course)
  - F can be less regular, but has to be coercive and lower semicontinuous.

Comparing the two methods

- The classical method
  - works by solving the EL equations
  - finds all critical points
  - needs F to be reasonably regular
- The direct method
  - works directly with the function, not the EL equations (Hence the name!)
  - finds only minima, not all critical points (There are direct methods for finding other critical points too, but those are beyond the scope of this course)
  - F can be less regular, but has to be coercive and lower semicontinuous.

The basic features of the two methods are the same in the calculus of variations too.

Comparison of the methods in the case of the calculus of variations

But there are some differences as well. For finding minima of a function, the EL equations are algebraic equations. In the Calculus of Variations, the EL equations are ODE, system of ODEs, PDE or a system of PDEs.

Comparison of the methods in the case of the calculus of variations

But there are some differences as well. For finding minima of a function, the EL equations are algebraic equations. In the Calculus of Variations, the EL equations are ODE, system of ODEs, PDE or a system of PDEs.

If they are PDEs or system of PDEs, even proving existence of a solution (let alone characterizing all solutions!) directly is hard!

Comparison of the methods in the case of the calculus of variations

But there are some differences as well. For finding minima of a function, the EL equations are algebraic equations. In the Calculus of Variations, the EL equations are ODE, system of ODEs, PDE or a system of PDEs.

If they are PDEs or system of PDEs, even proving existence of a solution (let alone characterizing all solutions!) directly is hard!

In fact, PDE theory is one of the main applications of the Calculus of Variations! Conversely, variational methods are among the most important tools in PDE theory.

If a PDE appears as the EL equation of some functional (which by the way is often the case), we usually prove existence of a solution by finding a critical point for the functional by direct methods, precisely going in the reverse direction as compared to the classical methods.

Calculus of Variation: The problem

We now want to pass from functions to functionals. Let us state our model problem.

Calculus of Variation: The problem

We now want to pass from functions to functionals. Let us state our model problem.

From functions to functionals

Let $\Omega \subset \mathbb{R}^n$ open, bounded, smooth. $n, N \geq 1$ are integers. Let $\mathcal{A}$ be a given class of functions $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a given function. Consider the following minimization problem

$$\inf \left\{ I(u) := \int_\Omega f\left(x, u(x), Du(x)\right)\, \mathrm{d}x : u \in \mathcal{A} \right\} \qquad \text{(P)}$$

Calculus of Variation: The problem

We now want to pass from functions to functionals. Let us state our model problem.

From functions to functionals

Let $\Omega \subset \mathbb{R}^n$ open, bounded, smooth. $n, N \geq 1$ are integers. Let $\mathcal{A}$ be a given class of functions $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a given function. Consider the following minimization problem

$$\inf \left\{ I(u) := \int_{\Omega} f\left(x, u(x), Du(x)\right)\, \mathrm{d}x : u \in \mathcal{A} \right\} \qquad \text{(P)}$$

- The integral functional $I(u)$ is called the **Lagrangian**
- The integral f is called the **Lagrangian density**
- The class $\mathcal{A}$ is called the **class of admissible functions**.

Calculus of Variation: The problem

We now want to pass from functions to functionals. Let us state our model problem.

From functions to functionals

Let $\Omega \subset \mathbb{R}^n$ open, bounded, smooth. $n, N \geq 1$ are integers. Let $\mathcal{A}$ be a given class of functions $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a given function. Consider the following minimization problem

$$\inf \left\{ I(u) := \int_{\Omega} f\left(x, u(x), Du(x)\right)\ \mathrm{d}x : u \in \mathcal{A} \right\} \qquad \text{(P)}$$

- The integral functional $I(u)$ is called the **Lagrangian**
- The integral f is called the **Lagrangian density**
- The class $\mathcal{A}$ is called the **class of admissible functions**.

The Lagrangian can depend on higher order derivatives of u. Those however are somewhat rare, though notable exceptions exist (e.g. Polyharmonic maps).

Isoperimetric problem

Perhaps the oldest known problem in the calculus of variations is the **isoperimetric problem**, which is just the **isoperimetric inequality** in dimension two.
The problem is to find *the(?)* geometric figure which has **the largest area with a fixed perimeter**.

The fact that the **circle** has this property is probably known since antiquity in many cultures around the world, including Greece, Egypt, India, Babylon, China etc. In Europe, it was traditionally known as the Dido problem.
Around 200 BCE, Zenodorous proved the inequality for polygons.
Archimedes, **Pappus**, **Euler**, **Galileo**, **Legendre**, **Riccati**, **Steiner**......

The first proof that agrees with modern standards is due to **Weierstrass**.

Blaschke, **Carathéodory**, **Frobenius**, **Hurwitz**, **Lebesgue**, **Liebmann**, **Minkowski**, **H.A. Schwarz**, **Sturm**, **Tonelli** among others.....

Brief History: Seventeenth to Twentieth Century

Seventeenth century and the Brachistochrone

- **Fermat (1662)** - geometric optics,
- **Newton (1685)** and **Huygens (1691)** - bodies moving through a fluid,
- **Gallileo (1638)** formulated the **Brachistochrone problem**, solved by **John Bernoulli (1696)**, **James Bernoulli**, **Newton** and **Leibnitz**.

Euler and **Lagrange** introduced what is now known as the **Euler-Lagrange equation**.

Bliss, **Bolza**, **Carathéodory**, **Clebsch**, **Hahn**, **Hamilton**, **Hilbert**, **Kneser**, **Jacobi**, **Legendre**, **Mayer**, **Weierstrass** and many many others.....

Brief History: Seventeenth to Twentieth Century

Nineteenth century and the Dirichlet integral

Dirichlet, **Gauss**, **Thompson** and **Riemann**...

Hilbert solved the problem, extending works of **Lebesgue** and **Tonelli**.

This problem inspired the development of most of modern analysis, namely functional analysis, measure theory, distribution theory, Sobolev spaces, partial differential equations.

Brief History: Seventeenth to Twentieth Century

Minimal surfaces

This is another central problem which has inspired a lot of analysis, including subjects like geometric measure theory.

Lagrange first formulated the problem in 1762.

However, this is often called **Plateau's problem** in honor of the Belgian physicist **Joseph Plateau**, whose experiments with soap films and his empirical 'Plateau's laws' influenced the status of the problem considerably.

Ampère, **Beltrami**, **Bernstein**, **Bonnet**, **Catalan**, **Darboux**, **Enneper**, **Haar**, **Korn**, **Legendre**, **Lie**, **Meusnier**, **Monge**, **Müntz**, **Riemann**, **H.A. Schwarz**, **Serret**, **Weierstrass**, **Weingarten** and others.

Douglas and **Rado** finally solved the problem in 1930. Douglas was awarded the *fields medal* for it in 1936!

Courant, **Leray**, **Mac Shane**, **Morrey**, **Morse**, **Tonelli**...

Now we are going to give a few classical examples, almost all of which were instrumental in driving the early research in the calculus of variations and paved the way for later developments.

- Fermat's principle of least time: The basic variational principle in geometric optics.
- Newton's problem: Finding the surface of revolution which experiences least resistance when moving through a fluid.
- Brachistochrone: Almost *the* iconic example in the classical calculus of variations.
- Principle of least action: Essentially the heart and soul of Newtonian mechanics.
- Minimal surface of revolution : The easier version of another iconic example: the minimal surface problem
- Dirichlet integral: The most celebrated and the protypical example in all of the calculus of variations.
- Minimal surfaces: Another star of the show! Almost as famous as the Dirichlet integral.
- Isoperimetric inequality By far the oldest variational problem to be noticed.

Fermat's principle of least time

Find the path of a light ray in a medium with nonconstant refractive index.

The ray follows the path of least time!

The variational problem:

$$\inf \left\{ I(u) := \int_a^b f\left(x, u(x), u'(x)\right) \, \mathrm{d}x : u(a) = \alpha, u(b) = \beta \right\}, \tag{5}$$

where $n = N = 1$ and the form of the Lagrangian is

$$f(x, u, \xi) = g(x, u)\sqrt{1 + \xi^2}.$$

Newton's optimal surface of revolution with least fluid resistance

Find the surface of revolution that experiences the least resistance while moving through a fluid.

The variational problem:

$$m = \inf \left\{ I(u) := \int_a^b f\left(u(x), u'(x)\right) \, \mathrm{d}x : u(a) = \alpha, u(b) = \beta \right\}, \tag{6}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = 2\pi u \left(\frac{\xi^3}{1 + \xi^2} \right).$$

Brachistochrone

Find the quickest path between two points for a point mass moving under gravity.

Brachistochrone

Find the quickest path between two points for a point mass moving under gravity.

Let one of the points be the origin $(0, 0) \subset \mathbb{R}^2$ and the other point is $(b, -\beta) \subset \mathbb{R}^2$ with $b, \beta > 0$. Gravity is acting downwards in the negative y-axis and the path is expressed as $(x, -u(x))$ with $0 \leq x \leq b$.

Brachistochrone

Find the quickest path between two points for a point mass moving under gravity.

Let one of the points be the origin $(0,0) \subset \mathbb{R}^2$ and the other point is $(b,-\beta) \subset \mathbb{R}^2$ with $b, \beta > 0$. Gravity is acting downwards in the negative y-axis and the path is expressed as $(x, -u(x))$ with $0 \leq x \leq b$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_a^b f\left(u(x), u'(x)\right) \, \mathrm{d}x : u \in \mathcal{A} \right\}, \qquad (7)$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x,u,\xi) = f(u,\xi) = \sqrt{\left(\frac{1+\xi^2}{2gu}\right)}.$$

Brachistochrone

Find the quickest path between two points for a point mass moving under gravity.

Let one of the points be the origin $(0,0) \subset \mathbb{R}^2$ and the other point is $(b, -\beta) \subset \mathbb{R}^2$ with $b, \beta > 0$. Gravity is acting downwards in the negative y-axis and the path is expressed as $(x, -u(x))$ with $0 \leq x \leq b$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_a^b f\left(u(x), u'(x)\right)\, \mathrm{d}x : u \in \mathcal{A} \right\}, \tag{7}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = \sqrt{\left(\frac{1+\xi^2}{2gu}\right)}.$$

The class of admissible paths is

$$\mathcal{A} := \left\{ \begin{aligned} u \in C^1\left([0,b]\right) : u(0) = 0,\ u(b) = \beta \\ \text{and } u(x) > 0 \text{ for all } x \in (0,b] \end{aligned} \right\}.$$

Brachistochrone

Find the quickest path between two points for a point mass moving under gravity.

Let one of the points be the origin $(0,0) \subset \mathbb{R}^2$ and the other point is $(b, -\beta) \subset \mathbb{R}^2$ with $b, \beta > 0$. Gravity is acting downwards in the negative y-axis and the path is expressed as $(x, -u(x))$ with $0 \leq x \leq b$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_a^b f\left(u(x), u'(x)\right) \, \mathrm{d}x : u \in \mathcal{A} \right\}, \qquad (7)$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = \sqrt{\left(\frac{1+\xi^2}{2gu}\right)}.$$

The class of admissible paths is

$$\mathcal{A} := \left\{ \begin{aligned} u \in C^1([0,b]) : u(0) = 0, \; u(b) = \beta \\ \text{and } u(x) > 0 \text{ for all } x \in (0, b] \end{aligned} \right\}.$$

The solution is called a **Cycloid**, which is also a **Tautochrone**.

Principle of least action: mechanics of system of point masses

Find the configuration of M point masses moving under a potential at time T.

Principle of least action: mechanics of system of point masses

Find the configuration of M point masses moving under a potential at time T.

Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \leq i \leq M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t. The **potential energy** function for the configuration $u(t)$ is a given function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$.

Principle of least action: mechanics of system of point masses

Find the configuration of M point masses moving under a potential at time T.

Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \leq i \leq M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t.

The **potential energy** function for the configuration $u(t)$ is a given function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_0^T f\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t : u(0) = u_0, \dot{u}(0) = v_0 \right\}, \tag{8}$$

where $n = 1, N = 3M$, u_0, v_0 given and the form of the Lagrangian density is

$$f(x, u, \xi) = T\left(\xi\right) - U\left(t, u(t)\right). \qquad \text{(usually called } \textbf{action} \text{)}$$

Principle of least action: mechanics of system of point masses

Find the configuration of M point masses moving under a potential at time T.

Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \leq i \leq M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t.

The **potential energy** function for the configuration $u(t)$ is a given function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_0^T f\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t : u(0) = u_0, \dot{u}(0) = v_0 \right\}, \tag{8}$$

where $n = 1, N = 3M$, u_0, v_0 given and the form of the Lagrangian density is

$$f(x, u, \xi) = T\left(\xi\right) - U\left(t, u(t)\right). \qquad \text{(usually called } \textbf{action} \text{)}$$

Here T is the **kinetic energy** and is given by

$$T\left(\xi\right) := \frac{1}{2} \sum_{i=1}^{M} m_i \xi_i^2.$$

Minimal surface of revolution

Determine the one with minimal area among all surfaces of revolution of the form

$$v(x, y) = (x, u(x)\cos y, u(x)\sin y)$$

with fixed end points $u(a) = \alpha, u(b) = \beta$.
Here $n = N = 1$ and the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = 2\pi u\sqrt{1 + \xi^2}$$

and the variational problem is

$$\inf \left\{ I(u) = \int_a^b f(u(x), u'(x))\, \mathrm{d}x : u(a) = \alpha, u(b) = \beta, u > 0 \right\} = m.$$

The solutions are called **Catenoids**.

Dirichlet integral

Arguably the most celebrated problem in all of the calculus of variations. We have here $n > 1, N = 1$ and

$$\inf \left\{ I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x : u = u_0 \text{ on } \partial\Omega \right\} = m.$$

The Euler-Lagrange equation is nothing other than the **Laplace equation**, namely

$$\Delta u = 0.$$

A generalized version of this is the p-**Dirichlet integral**,

$$\inf \left\{ I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, \mathrm{d}x : u = u_0 \text{ on } \partial\Omega \right\} = m,$$

where $1 < p < \infty$. The Euler-Lagrange equation is the p-**Laplace equation**, i.e.

$$\Delta_p u := \operatorname{div} \left(|\nabla u|^{(p-2)} \nabla u \right) = 0.$$

This is quasilinear if $p \neq 2$, degenerate elliptic if $p > 2$ and singular elliptic if $1 < p < 2$.

Minimal surfaces

The question is
to find among all surfaces $\Sigma \subset \mathbb{R}^3$ (or more generally in $\mathbb{R}^{n+1}$, $n \geq 2$)
with prescribed boundary, $\partial\Sigma = \Gamma$,
where Γ is a simple closed curve,
one that is of minimal area.

A variant of this problem is known as *Plateau problem*.

One can experimentally realize such surfaces by dipping a wire loop into soapy water; the surface obtained when pulling the wire out from the water is a minimal surface.

The precise formulation of the problem depends on the kind of surfaces that we are considering. We have seen above how to write the problem for minimal surfaces of revolution. We now formulate the problem for more general surfaces.

Nonparametric surfaces

We consider (hyper) surfaces of the form

$$\Sigma = \left\{ v\left(x\right) = \left(x, u\left(x\right)\right) \in \mathbb{R}^{n+1} : x \in \overline{\Omega} \right\}$$

with $u : \overline{\Omega} \to \mathbb{R}$ and where $\Omega \subset \mathbb{R}^n$ is a bounded connected open set.

These surfaces are therefore graphs of functions.

The fact that $\partial\Sigma$ is a preassigned curve, Γ, reads now as $u = u_0$ on $\partial\Omega$, where u_0 is a given function. The area of such a surface is given by

$$\text{Area}\left(\Sigma\right) = I\left(u\right) = \int_{\Omega} f\left(\nabla u\left(x\right)\right) dx$$

where, for $\xi \in \mathbb{R}^n$, we have set

$$f\left(\xi\right) = \sqrt{1 + \left|\xi\right|^2}.$$

Minimal surfaces: Nonparametric surfaces

The problem is then written in the usual form

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x))\, dx : u = u_0 \text{ on } \partial\Omega \right\},$$

with

$$f(\xi) = \sqrt{1 + |\xi|^2} \qquad \text{for } \xi \in \mathbb{R}^n.$$

Associated with (P) we have the so-called *minimal surface equation*

$$(E) \quad \mathrm{M}u \equiv \left(1 + |\nabla u|^2\right) \Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

which is the equation that any minimizer u of (P) should satisfy. In geometrical terms, this equation just expresses the fact that the corresponding surface Σ has everywhere vanishing *mean curvature*.

Parametric surfaces

Nonparametric surfaces are clearly too restrictive from the geometrical point of view and one is led to consider *parametric surfaces*. These are sets $\Sigma \subset \mathbb{R}^{n+1}$ so that there exist a connected open set $\Omega \subset \mathbb{R}^n$ and a map $v : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that

$$\Sigma = v\left(\overline{\Omega}\right) = \left\{ v\left(x\right) : x \in \overline{\Omega} \right\}.$$

For example, when $n = 2$ and $v = v\left(x_1, x_2\right) \in \mathbb{R}^3$, if we denote by $v_{x_1} \times v_{x_2}$ the normal to the surface (where $a \times b$ stands for the vectorial product of $a, b \in \mathbb{R}^3$ and $v_{x_1} = \partial v / \partial x_1$, $v_{x_2} = \partial v / \partial x_2$) we find that the area is given by

$$\text{Area}\left(\Sigma\right) = J\left(v\right) = \iint_{\Omega} \left| v_{x_1} \times v_{x_2} \right| dx_1 dx_2 .$$

In terms of the notations introduced at the beginning of the present section we have $n = 2$ and $N = 3$.

Isoperimetric inequality in dimension two

Let $A \subset \mathbb{R}^2$ be a bounded open set whose boundary, ∂A, is a sufficiently regular simple closed curve. Denote by $L(\partial A)$ the length of the boundary and by $M(A)$ the measure (the area) of A. The isoperimetric inequality states that

$$[L(\partial A)]^2 - 4\pi M(A) \geq 0.$$

Equality holds if and only if A is a disk (i.e. ∂A is a circle).

Isoperimetric inequality in any dimension

For open sets $A \subset \mathbb{R}^n$ with sufficiently regular boundary, ∂A, and it reads as

$$[L(\partial A)]^n - n^n \omega_n [M(A)]^{n-1} \geq 0$$

where ω_n is the measure of the unit ball of $\mathbb{R}^n$, $M(A)$ stands for the measure of A and $L(\partial A)$ for the $(n-1)$ measure of ∂A. Moreover, if A is sufficiently regular (for example, convex), there is equality if and only if A is a ball.

Isoperimetric inequality

We can rewrite this into our formalism (here $n = 1$ and $N = 2$) by parametrizing the curve

$$\partial A = \left\{ u(x) = \left(u^1(x), u^2(x) \right) : x \in [a, b] \right\}$$

and setting

$$L(\partial A) = L(u) = \int_a^b \sqrt{\left((u^1)' \right)^2 + \left((u^2)' \right)^2},$$

$$M(A) = M(u) = \frac{1}{2} \int_a^b \left(u^1 \left(u^2 \right)' - u^2 \left(u^1 \right)' \right) = \int_a^b u^1 \left(u^2 \right)' .$$

The problem is then to show that

$$(P) \quad \inf \left\{ L(u) : M(u) = 1; \; u(a) = u(b) \right\} = 2\sqrt{\pi}.$$

Thank you

Questions?