Introduction to the Calculus of Variations
Lecture Notes
Lecture 7-8

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Chapter 1

Classical Methods

1.1 Second Variation

So far in this chapter we were concerned with **any critical point**. Now we want to investigate necessary and sufficient conditions for a given critical point to be a minimizer of the functional. We begin with a simple result which calculates the second variation and gives a necessary criterion a given critical point to be a minimizer in terms of the second variation.

Theorem 1 (Second Variation)**.** *Let $f = f(t, u, \xi) \in C^3\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$ be given and $X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$. Consider the problem*

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

If $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ is a minimizer for (P), then the following integral

$$\int_a^b \left[\langle f_{uu}(t, \bar{u}, \dot{\bar{u}})\, \psi, \psi \rangle + 2 \left\langle f_{u\xi}(t, \bar{u}, \dot{\bar{u}})\, \psi, \dot{\psi} \right\rangle + \left\langle f_{\xi\xi}(t, \bar{u}, \dot{\bar{u}})\, \dot{\psi}, \dot{\psi} \right\rangle \right] \tag{1.1}$$

*is **nonnegative** for any $\psi \in C_c^1\left([a, b]; \mathbb{R}^N\right)$.*

Proof. As we did in deriving the EL equations, we take $\psi \in C_c^1\left([a, b]; \mathbb{R}^N\right)$. Thus for any $h \in \mathbb{R}$, we have $\bar{u} + h\psi \in X$. Now we define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(h) := I(\bar{u} + h\psi)$.

Then $g \in C^2(\mathbb{R})$ (Check!) and since $\bar{u}$ is a minimizer, g must have a local minima at 0. Thus we must have $g''(0) \geq 0$. But

$$g''(0) = \frac{d^2}{dh^2} \left[I(\bar{u} + h\psi) \right] \Big|_{h=0}.$$

The rest is a straight forward calculation. Note that since $f \in C^3$, thus in particular C^2 and thus $f_{u\xi}$, f_{uu} and $f_{\xi\xi}$ are all symmetric matrices. $\square$

Quadratic functional related to second variation

To understand the expression for the second variation better, we integrate by parts in the mixed term and obtain

$$2\int_a^b \left\langle f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\psi,\dot{\psi}\right\rangle = -\int_a^b \left\langle \frac{d}{dt}\left[f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\right]\psi,\psi\right\rangle .$$

Note that here again we used the fact that $f_{u\xi}$ takes values in the space of symmetric matrices. In view of this, we can rewrite the expression (1.1) as

$$\int_a^b \left[\left\langle\left[f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\right]\psi,\psi\right\rangle+\left\langle f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\dot{\psi},\dot{\psi}\right\rangle\right].$$

The important point here is that the matrix $f_{\xi\xi}$ plays the dominant role here in determining whether this quadratic form will be nonnegative or not.

The heuristic argument is, since ψ vanishes at the boundary, we have a Poincaré inequality. Roughly, since the function vanishes at the boundary, the value of the function itself can not be large while keeping its derivative small, since it has climb up from zero to the high values. But the converse is quite possible! The function can be small with large derivative! Why? It can oscillate a lot!

We now formalize the heuristic argument.

Lemma 2. *If the following inequality*

$$\int_a^b \left[\left\langle\left[f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\right]\psi,\psi\right\rangle+\left\langle f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\dot{\psi},\dot{\psi}\right\rangle\right]\ \mathrm{d}t\ \geq\ 0,$$

holds for every $\psi\in C_c^1\left([a,b];\mathbb{R}^N\right)$, then the matrix $f_{\xi\xi}\left(t,\bar{u},\dot{\bar{u}}\right)$ is nonnegative definite for every $t\in(a,b)$.

Proof. If $f_{\xi\xi}<0$ for some $t_0\in(a,b)$, this means there exist a $\zeta\in\mathbb{R}^N$ and $\beta>0$ such that

$$\left\langle f_{\xi\xi}\left(t_0,\bar{u}\left(t_0\right),\dot{\bar{u}}\left(t_0\right)\right)\zeta,\zeta\right\rangle<-\beta.$$

By continuity of $f_{\xi\xi}$, we can assume there exists $\alpha>0$ such that $a<t_0-\alpha<t_0+\alpha<b$ and we have

$$\left\langle f_{\xi\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\zeta,\zeta\right\rangle<-\beta \qquad \text{for all } t\in[t_0-\alpha,t_0+\alpha].$$

Choose

$$\psi\left(t\right)=\begin{cases}\alpha\sin^2\left[\dfrac{\pi\left(t-t_0\right)}{\alpha}\right]\zeta & \text{if } t\in[t_0-\alpha,t_0+\alpha]\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\psi \in C_c^1\left([a,b];\mathbb{R}^N\right)$ and plugging it, we obtain

$$\pi^2 \int_{t_0-\alpha}^{t_0+\alpha} \sin^2\left[\frac{2\pi\left(t-t_0\right)}{\alpha}\right]\left\langle f_{\xi\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\zeta,\zeta\right\rangle\ \mathrm{d}t$$
$$+\alpha^2\int_{t_0-\alpha}^{t_0+\alpha}\sin^4\left[\frac{\pi\left(t-t_0\right)}{\alpha}\right]\left\langle\left[f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\right]\zeta,\zeta\right\rangle\ \mathrm{d}t\geq 0$$

But this implies

$$2M\alpha^3-2\beta\pi^2\alpha\geq 0,$$

where

$$M=\max_{t\in[t_0-\alpha,t_0+\alpha]}\left|\left\langle\left[f_{uu}\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}f_{u\xi}\left(t,\bar{u},\dot{\bar{u}}\right)\right]\zeta,\zeta\right\rangle\right|.$$

But this means

$$\beta\leq\frac{M}{\pi^2}\alpha^2,$$

which we can easily contradict by letting $\alpha\to 0$. So we deduce

$$\left\langle f_{\xi\xi}\left(t,\bar{u}\left(t\right),\dot{\bar{u}}\left(t\right)\right)\zeta,\zeta\right\rangle\geq 0\qquad\text{for all }\zeta\in\mathbb{R}^N,\ \text{for all }t\in(a,b).$$

This proves the lemma. □

This condition is known as the **Legendre condition**. This is implied by **convexity** of the map $\xi\mapsto f\left(t,u,\xi\right)$. If $n>1$, $N>1$, then the corresponding condition is called the **Legendre-Hadamard condition**.

$$\left\langle f_{\xi\xi}\left(x,\bar{u}\left(x\right),D\bar{u}\left(x\right)\right)a\otimes b,a\otimes b\right\rangle\geq 0$$

for all $a\in\mathbb{R}^n$, $b\in\mathbb{R}^N$ and for all $x\in\Omega$. This is **weaker** than the Legendre condition in that case, which would read

$$\left\langle f_{\xi\xi}\left(x,\bar{u}\left(x\right),D\bar{u}\left(x\right)\right)\xi,\xi\right\rangle\geq 0$$

for all $\xi\in\mathbb{R}^{n\times N}$ and for all $x\in\Omega$. Another way to see that these two conditions are different in the vectorial case $(n,N>1)$ is to notice that the Legendre-Hadamard is implied by convexity only along rank one matrices, which is in general significantly weaker than convexity.

Towards a sufficient condition

Possible candidate for sufficiency Can $f_{\xi\xi}\geq 0$ be a sufficient condition? **Clearly not!** Just think of $f(x)=x^3$, $x\in\mathbb{R}$. $x=0$ is a critical point and the second derivative vanishes, but it is not a minima! Then, can $f_{\xi\xi}>0$, i.e. positive definite instead of nonnegative definite, be a sufficient condition? This looks more promising, as this would be enough for finding minima of functionals. However, somewhat surprisingly, the answer is still **No!**

Understanding the trouble The reason is that the condition is purely local, whereas being a minimizer is not really a local property. We go back to geodesics. Think of the unit sphere in $\mathbb{R}^3$ centered at the origin and consider the points $A = (1,0,0)$, $B = (0,1,0)$ and $C = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$. All three points lie on the circle $\left\{(x,y,0) : x^2 + y^2 = 1\right\}$, which being a great circle is a geodesic on the sphere. Now, the part of the circle going from A to B is a minimizing path with lenth $\pi/2$ and so is the part of the circle going from B to C, which has length $3\pi/4$. However, clearly the part of the circle going from A to C can not be minimizing, since its length is $\pi/2 + 3\pi/4 = 5\pi/4$, whereas the part of the circle going from C to A is definitely shorter, with length $2\pi - 5\pi/4 = 3\pi/4$.

Jacobi theory and Legendre method

We now consider the second variation itself as an integral functional

$$J[\psi] := \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi}\right\rangle + \langle Q\psi, \psi\rangle\right]\ \mathrm{d}t, \qquad \psi \in C^1, \psi(a) = \psi(b) = 0,$$

where

$$P := f_{\xi\xi}(t, \bar{u}, \dot{\bar{u}}) \qquad \text{and} \qquad Q := \left[f_{uu}(t, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt} f_{u\xi}(t, \bar{u}, \dot{\bar{u}})\right]$$

Note that it is not difficult to establish that if

$$J[\psi] > c\int_a^b \left|\dot{\psi}\right|^2, \qquad \text{for all } \psi \in C^1, \psi \not\equiv 0 \text{ with } \psi(a) = \psi(b) = 0, \tag{1.2}$$

for some $c > 0$, then $\bar{u}$ is a minimizer. (See the problem sheet for a detailed proof). Note that

$$J[\psi] > 0, \qquad \text{for all } \psi \in C^1, \psi \not\equiv 0 \text{ with } \psi(a) = \psi(b) = 0,$$

is not sufficient, as can be seen in the following example.

Example 3. *The Lagrangian*

$$I[u] = \int_{-1}^1 \left[t^2 (\dot{u}(t))^2 + t(\dot{u}(t))^3\right]\ \mathrm{d}t,$$

has $u \equiv 0$ as an extremal where the second variation is positive for every non-trivial test function, but $u \equiv 0$ is not a minimizer. Indeed, at $u \equiv 0$, the second variation is

$$J[\psi] = \int_{-1}^1 t^2 \left(\dot{\psi}(t)\right)^2\ \mathrm{d}t,$$

which is clearly positive for every $\psi \in C^1$, $\psi \not\equiv 0$ with $\psi(a) = \psi(b) = 0$. Now consider the family of functions

$$u_\varepsilon(t) := \begin{cases} \varepsilon\left(\frac{3}{4}\varepsilon + t\right) & \text{for } -\frac{3}{4}\varepsilon \le t \le 0, \\ \varepsilon\left(\frac{3}{4}\varepsilon - t\right) & \text{for } 0 \le t \le \frac{3}{4}\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Now we have $I[u_\varepsilon] = -\frac{9}{32}\varepsilon^5 < 0 = I[0]$. We can easily round off the corners of u_ε, making it C^1 and still keep the condition that the functional is strictly negative on these functions. Moreover, clearly these modified C^1 functions converge in C^1 to $u \equiv 0$ as $\varepsilon \to 0$. So $u \equiv 0$ does not minimize the functional among C^1 functions in its C^1 neighborhood.

However, $P > 0$, i.e. positive definiteness of P for all $t \in (a, b)$ is not enough to obtain (1.2). So what other condition is needed to ensure this? To find out, Legendre wanted to 'complete the square' by adding a **null Lagrangian**, i.e. *integral functionals which always equate to zero irrespective of the argument in the class of admissible functions.*

Legendre method Let W be an arbitrary differentiable symmetric matrix. Then

$$0 = \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\, dt \qquad \text{for all } \psi \text{ with } \psi(a) = \psi(b) = 0.$$

Thus

$$\frac{d}{dt}\left[\langle W\psi, \psi\rangle\right] \text{ is a null lagrangian for any } W.$$

Hence adding such a term does not alter the value of $J[\psi]$. So we get

$$\begin{aligned} J[\psi] &= J[\psi] + \int_a^b \frac{d}{dt}\left[\langle W\psi, \psi\rangle\right]\, dt \\ &= \int_a^b \left[\left\langle P\dot\psi, \dot\psi\right\rangle + 2\left\langle W\psi, \dot\psi\right\rangle + \left\langle \left(Q + \dot W\right)\psi, \psi\right\rangle\right]\, dt \end{aligned}$$

When can we make this a perfect square?

Riccati equation

Proposition 4. *Suppose W is a solution of the following* matrix Riccati equation*,*

$$\dot W = -Q + W P^{-1} W. \tag{1.3}$$

Then we have

$$\left[\left\langle P\dot\psi, \dot\psi\right\rangle + 2\left\langle W\psi, \dot\psi\right\rangle + \left\langle \left(Q + \dot W\right)\psi, \psi\right\rangle\right] = \left|P^{\frac{1}{2}}\dot\psi + P^{-\frac{1}{2}}W\psi\right|^2.$$

Remark 5. *Here $P^{\frac{1}{2}}$ is the square root of P. Note that since P is symmetric and positive definite, the square root $P^{\frac{1}{2}}$ is well defined and is itself symmetric and positive definite.*

Proof. The proof is elementary calculation. Indeed, we have

$$\begin{aligned}
\left|P^{\frac{1}{2}}\dot{\psi} + P^{-\frac{1}{2}}W\psi\right|^2 &= \left\langle P^{\frac{1}{2}}\dot{\psi} + P^{-\frac{1}{2}}W\psi, P^{\frac{1}{2}}\dot{\psi} + P^{-\frac{1}{2}}W\psi\right\rangle \\
&= \left\langle P^{\frac{1}{2}}\dot{\psi}, P^{\frac{1}{2}}\dot{\psi}\right\rangle + \left\langle P^{\frac{1}{2}}\dot{\psi}, P^{-\frac{1}{2}}W\psi\right\rangle + \left\langle P^{-\frac{1}{2}}W\psi, P^{\frac{1}{2}}\dot{\psi}\right\rangle \\
&\qquad + \left\langle P^{-\frac{1}{2}}W\psi, P^{-\frac{1}{2}}W\psi\right\rangle \\
&= \left\langle P\dot{\psi}, \dot{\psi}\right\rangle + \left\langle \dot{\psi}, W\psi\right\rangle + \left\langle W\psi, \dot{\psi}\right\rangle + \left\langle P^{-1}W\psi, W\psi\right\rangle \\
&= \left\langle P\dot{\psi}, \dot{\psi}\right\rangle + 2\left\langle W\psi, \dot{\psi}\right\rangle + \left\langle WP^{-1}W\psi, \psi\right\rangle \\
&\overset{(1.3)}{=} \left\langle P\dot{\psi}, \dot{\psi}\right\rangle + 2\left\langle W\psi, \dot{\psi}\right\rangle + \left\langle \left(Q + \dot{W}\right)\psi, \psi\right\rangle.
\end{aligned}$$

This completes the proof. □

Now, to solve the Riccati equation

$$\dot{W} = -Q + WP^{-1}W,$$

let us substitute

$$W = -P\dot{\Psi}\Psi^{-1}.$$

Plugging it in the Riccati equation, we obtain

$$\frac{d}{dt}\left(P\dot{\Psi}\right) = Q\Psi. \tag{1.4}$$

Any solution Ψ of the above equation would furnish a solution W of the Riccati equation if Ψ is **invertible.**

Jacobi equation and Jacobi fields However, the equation above has another nice interpretation. We again consider the second variation itself as an integral functional

$$J[\psi] := \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi}\right\rangle + \langle Q\psi, \psi\rangle\right] \mathrm{d}t, \qquad \psi \in C^1, \psi(a) = \psi(b) = 0.$$

The Euler-Lagrange equation to this variational problem is

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi. \tag{1.5}$$

This is called the **Jacobi equation** and its solutions (for a given u) is called a **Jacobi field along** u**.** Clearly, this equation looks very much the same as equation (1.4), except that here the unknown ψ is $\mathbb{R}^N$-valued, whereas Ψ in

(1.4) is an $N \times N$ matrix-valued function. However, since the form of both the equations are the same, the matrix formed by a system of N linearly independent solutions of (1.5) would solve (1.4). In fact, any N solutions of (1.5) would have the same property. However, they being linearly independent would mean that Ψ is invertible as well and thus would furnish a solution of the matrix Ricatti equation. Thus, linear independence is a crucially important property in this context and we would do well to be very interested in determining if and when this fails. In accordance with this line of thinking, we define the notion of conjugate points.

Definition 6 (Conjugate points)**.** *Let Ψ be the matrix of N solutions of the Jacobi equation, i.e.*

$$\Psi := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}.$$

where $\psi_1, \ldots, \psi_N$ solves the Jacobi equation and satisfies

$$\Psi(a) = 0 \quad \text{and} \quad \dot{\Psi}(a) = \mathbb{I}_N.$$

A point $\bar{a} \in (a, b]$ is called a ***conjugate to the point*** *a or simply a* ***conjugate point of*** *a if we have*

$$\det \Psi(\bar{a}) = 0.$$

At this stage, it should be clear what we are trying to achieve. If there are no conjugate points to a in $(a, b]$ for $J[\psi]$, then there would not be one as well for the functional

$$J_c[\psi] := J[\psi] - c \int_a^b \left|\dot{\psi}\right|^2,$$

by continuous dependence of solutions to ODEs on parameters. But then the corresponding Ψ would be invertible for every $t \in (a, b]$ and thus would furnish a solution to the corresponding Riccati equation. This in turn would imply that $J_c[\psi] > 0$ for all nontrivial $\psi \in C_c^1((a, b))$, which is sufficient for $\bar{u}$ to be a minimizer.

Now our goal is to show that this is a sufficient condition for $\bar{u}$ to be a minimizer.

1.1.1 Sufficient condition for a minimizer

Theorem 7. *Let $f = f(t, u, \xi) \in C^3\left([a, b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$, $X = \left\{u \in C^1\left([a, b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$.*

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a, b]; \mathbb{R}^N\right)$ be a ***critical point*** *of I such that*

- $f_{\xi\xi}(t, \bar{u}(t), \dot{\bar{u}}(t))$ *is* ***positive definite*** *for every* $t \in [a, b]$,
- *there exists no point in* $(a, b]$ *which is* ***conjugate*** *to* a.

Then $\bar{u}$ *is a* ***minimizer*** *of* I.

Proof. We need to show that the hypotheses implies that there exists $c > 0$ such that

$$J[\psi] > c \int_a^b \left|\dot{\psi}\right|^2 \tag{1.6}$$

for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$, as this is sufficient for $\bar{u}$ to be a minimizer. We set

$$J_c[\psi] := J[\psi] - c \int_a^b \left|\dot{\psi}\right|^2.$$

Clearly, this has the same form as $J[\psi]$ with P replaced by

$$P_c := P - c\mathbb{I}_N.$$

So the corresponding Jacobi equation is

$$\frac{d}{dt}\left[(P - c\mathbb{I}_N)\dot{\psi}\right] = Q\psi.$$

Now, by continuous dependence of solutions to ODEs on parameters, the above Jacobi equation also does not have a point conjugate to a for small enough $c > 0$, since there is none, by hypothesis, for the equation

$$\frac{d}{dt}\left(P\dot{\psi}\right) = Q\psi.$$

Also as $f_{\xi\xi}$, i.e. P is positive definite, so must be P_c for $c > 0$ small enough. Thus, as we saw already, we can write

$$J_c[\psi] = \int_a^b \left|P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi\right|^2 \, \mathrm{d}t.$$

So if $J_c[\psi] \not> 0$, then we must have

$$P_c^{\frac{1}{2}}\dot{\psi} + P_c^{-\frac{1}{2}}W\psi = 0 \qquad \text{for all } t \in (a, b)$$

for some $\psi \in C^1([a, b])$ with $\psi(a) = 0 = \psi(b)$. But the above is the first order ODE

$$\dot{\psi} = -\left(P_c^{-1}W\right)\psi.$$

Since ψ satisfies the initial condition $\psi(a) = 0$, by uniqueness of solutions of ODE, we must have $\psi \equiv 0$. So $J_c[\psi] > 0$ for all $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = 0 = \psi(b)$. This proves (1.6). Thus $\bar{u}$ is a minimizer and this completes the proof. $\square$

1.1.2 Jacobi fields and conjugate points

Now we want to investigate how sharp the conditions are. More precisely, we wish to show that our sufficient conditions, especially about the absence of conjugate points are actually very close to being necessary. But before analyzing these questions more deeply, we need to show the relation between **conjugate points** and **zeros of Jacobi fields**.

Proposition 8. *Let a^* be a conjugate point of a. Then there exists a Jacobi field $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$.*

Proof. Since a^* is a conjugate point, $\det \Psi(a^*) = 0$. Thus, the rows of Ψ are linearly dependent at a^*. Hence, there exists a linear combination of rows of Ψ

$$\eta(t) = \sum_{i=1}^{N} \mu_i \psi_i(t)$$

which is not identically zero and satisfies

$$\eta(a) = 0 = \eta(a^*).$$

Since each ψ_i solves the Jacobi equation, so does η. □

Vanishing of Jacobi fields and the quadratic forms Now we show that the existence of a Jacobi field which vanishes at an interior point has an important consequence.

Proposition 9. *If $\eta \in C^1\left([a, a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, is a Jacobi field on $[a, a^*]$ such that $\eta(a) = 0 = \eta(a^*)$, then we have*

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \,\mathrm{d}t = 0.$$

Proof. Since $\eta(a) = 0 = \eta(a^*)$, we can integrate by parts to obtain

$$\int_a^{a^*} \left[\langle P\dot{\eta}, \dot{\eta}\rangle + \langle Q\eta, \eta\rangle\right] \,\mathrm{d}t = \int_a^{a^*} \left\langle \left[-\frac{d}{dt}\left(P\dot{\eta}\right) + Q\eta\right], \eta \right\rangle \,\mathrm{d}t.$$

But the expression in the bracket vanishes as η is a Jacobi field. □

1.1.3 Jacobi's necessary condition

Now we are ready to show that the absence of interior conjugate points is almost necessary for the existence of a minimizer. However, we show the theorem for minimization problems in a slightly larger class C^1_{piece}. This has the advantage that the proof is considerably simpler, but the price we have to pay is that we would need to use a regularity result that we have not seen yet. However, since we are going to see the regularity result soon anyway and as such both this proof and the regularity result are more in tune with the spirit of modern direct methods, it is instructive to follow this proof.

Theorem 10. *Let* $f = f(t, u, \xi) \in C^3\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\alpha, \beta \in \mathbb{R}^N$, $X_{piece} = \left\{u \in C^1_{piece}\left([a,b]\,; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$.

$$(P) \qquad \inf_{u \in X_{piece}} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2\left([a,b]\,; \mathbb{R}^N\right)$ *be a* ***minimizer*** *of* I *such that* $f_{\xi\xi}(t, \bar{u}(t), \dot{\bar{u}}(t))$ *is* ***positive definite*** *for every* $t \in [a,b]$. *Then there exists no point in* (a,b) *which is conjugate to* a.

Remark 11. *Conjugate points in* (a,b) *are called* ***interior conjugate points*** *for obvious reasons.*

Question: Did we obtain a necessary and sufficient condition? **NO!**

- $f_{\xi\xi}$ is **positive definite** for every $t \in [a,b]$ is an explicit assumption! Not a necessary condition. Only $f_{\xi\xi}$ **nonnegative definite** everywhere in $[a,b]$ is necessary.
- b not being conjugate to a is needed for sufficiency, but is not necessary.

Now we are ready to prove Jacobi's necessary condition theorem.

Proof. This boils down to proving that if P is positive definite and

$$J[\psi] = \int_a^b \left[\left\langle P\dot{\psi}, \dot{\psi}\right\rangle + \langle Q\psi, \psi\rangle\right]\ \mathrm{d}t \geq 0$$

for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, then exists no point in (a,b) which is conjugate to a.

Suppose, if possible, that $a^* \in (a,b)$ is a conjugate point of a. Then as we have just shown, this implies that there exists a Jacobi field $\eta \in C^1\left([a,a^*], \mathbb{R}^N\right)$, $\eta \not\equiv 0$, on $[a,a^*]$ such that $\eta(a) = 0 = \eta(a^*)$. Now we set

$$\eta^* = \begin{cases} \eta & \text{if } t \in [a,a^*] \\ 0 & \text{if } t \in [a^*,b] \end{cases}.$$

Clearly η^* is piecewise C^1. We shall prove that this actually is C^2.

Now since $J[\eta^*] = 0$ and $J[\psi] \geq 0$ for every $\psi \in C^1_{\text{piece}}([a,b])$ with $\psi(a) = 0 = \psi(b)$, η^* is a minimizer for J. Since P is positive definite, we shall soon see that this implies $\eta^* \in C^2$. Thus, $\dot{\eta^*}$ is continuous across a^* and thus

$$\dot{\eta^*}(a^*) = 0.$$

But η^* satisfies the Jacobi equation, which is a **second order** ODE and we have $\eta^*(a^*) = 0$ and $\dot{\eta^*}(a^*) = 0$. By **uniqueness of solutions of ODE**, this implies $\eta^* \equiv 0$, which is a contradiction. □

Now we would show the same for the usual minimization problem in the C^1 class, but **only for the case** $N = 1$. Things simplify considerably for this assumption and the proof still is not completely transparent. However, as we will see, the ideas in this proof are somewhat different. The advantage of the piecewise C^1 setting was that we could directly plug in the 'broken' jacobi field (the extension by zero of a nontrivial Jacobi field defined on a smaller interval). In the C^1 setting, we do not have that opportunity. However, the more regular setting on the other hand gives us the luxury of using the implicit function theorem and continuous dependence on parameters of solutions of ODEs.

Theorem 12. *Let* $f = f(t, u, \xi) \in C^3([a, b] \times \mathbb{R} \times \mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, *and*

$$X = \left\{u \in C^1([a, b]; \mathbb{R}) : u(a) = \alpha,\ u(b) = \beta\right\}.$$

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{d}t \right\} = m.$$

Let $\bar{u} \in X \cap C^2([a, b]; \mathbb{R})$ *be a* ***minimizer*** *of* I *such that* $f_{\xi\xi}(t, \bar{u}(t), \dot{\bar{u}}(t)) > 0$ *for every* $t \in [a, b]$. *Then there exists no point in* (a, b) *which is conjugate to* a.

Let us briefly explain the ideas. Instead of working with the functional J, we would consider a one-parameter family of functionals J_λ such that as λ varies, we move continuously from our functional J to another one for which it is plainly evident that there could be no conjugate points. Then we argue that as conjugate points are about existence of zeros of nontrivial Jacobi fields and Jacobi fields are solutions of ODEs, namely the Jacobi equations, they vary continuously with respect to the parameter λ and thus can not appear suddenly. This argument, in many different guises are used a lot in PDE theory and is generally called **method of continuity**. The more topologically oriented reader would recognize the method as a **'homotopy method'**, where J_λ is our homotopy from our functional J to another one which are much easier to analyze.

We begin with a lemma which analyzes the 'simpler functional'.

Lemma 13. *The functional*

$$\tilde{I}(u) := \frac{1}{2} \int_a^b |\dot{u}(t)|^2\ \mathrm{d}t$$

has no conjugate points in $(a, b]$.

Proof. This is really elementary. If there is a conjugate point, by Proposition 8, we must have a nontrivial Jacobi field vanishing at the conjugate point. But, clearly $f_{\xi\xi}$ is the identity matrix and all other terms in the quadratic form is zero matrix. Thus the quadratic form for this functional is

$$\mathcal{Q}(\psi) = \int_a^b \left|\dot{\psi}(t)\right|^2\ \mathrm{d}t$$

and hence the Jacobi equation for this functional is

$$\ddot{\psi} = 0.$$

But then any Jacobi field vanishing at a must necessarily be of the form

$$\psi = t - a.$$

So this clearly can not vanish at any point in $(a, b]$. This contradiction establishes the lemma. $\square$

Now we are ready to prove our theorem.

Proof. **(of Theorem 12)** For $0 \leq \lambda \leq 1$, we consider the functional

$$I_\lambda (u) := \lambda I (u) + \frac{1}{2} (1 - \lambda) \int_a^b |\dot{u} (t)|^2 \, \mathrm{d}t.$$

Clearly, this is a homotopy between $I_0 = I$ and $I_1 = \tilde{I}$. The corresponding quadratic form is

$$J_\lambda (\psi) = \lambda J (\psi) + (1 - \lambda) \mathcal{Q} (\psi) := \lambda J (\psi) + (1 - \lambda) \int_a^b \left| \dot{\psi} (t) \right|^2 \, \mathrm{d}t.$$

Since $\bar{u}$ is a minimizer for I, we have $J (\psi) \geq 0$ for every $\psi \in C^1 ([a, b])$ with $\psi (a) = \psi (b) = 0$. Thus, clearly,

$$J_\lambda (\psi) > 0 \qquad \text{for } \lambda \in [0, 1)$$

for every $\psi \in C^1 ([a, b])$, $\psi \not\equiv 0$ with $\psi (a) = \psi (b) = 0$ and $J_\lambda (\psi) \geq 0$ for any $\lambda \in [0, 1]$ for every $\psi \in C^1 ([a, b])$ with $\psi (a) = \psi (b) = 0$. The Jacobi equation is

$$\frac{d}{dt} \left([\lambda P + (1 - \lambda)] \dot{\psi} \right) = \lambda Q \psi. \tag{1.7}$$

Since $P > 0$, so is $\lambda P + (1 - \lambda)$ for any $\lambda \in [0, 1]$. Now suppose, if possible, $a^* \in (a, b)$ is a conjugate point for I. By Proposition 8, this implies there exists a nontrivial Jacobi field for I vanishing at a^*. More precisely, this means there exists a $\eta \not\equiv 0$ with $\eta(a) = 0 = \eta(a^*)$ solving

$$\frac{d}{dt} (P \dot{\eta}) = Q \eta.$$

Now, there exists a solution $\psi = \psi (t, \lambda)$ of (1.7) which

(1) depends continuously on λ and has continuous partial derivative with respect to both t and λ in $[a, b] \times [0, 1)$, [1]

[1] This follows from the fact that (1.7) is equivalent to

$$[\lambda P + (1 - \lambda)] \ddot{\psi} + \left(\frac{d}{dt} [\lambda P + (1 - \lambda)] \right) \dot{\psi} = \lambda Q \psi.$$

(2) $\psi(t, \lambda_0) \not\equiv 0$ for any $\lambda_0 \in [0, 1]$

(3) and satisfies

$$\psi(t, 1) = \eta(t) \qquad \text{and} \qquad \psi(t, 0) = t - a.$$

Also note that if $\psi(t_0, \lambda_0) = 0$ for any $(t_0, \lambda_0) \in [a, b] \times [0, 1]$, then

$$\frac{\partial \psi}{\partial t}(t_0, \lambda_0) \neq 0, \tag{1.8}$$

since otherwise by uniqueness of solutions for second order ODEs, we would have $\psi(t, \lambda_0) = 0$ for all $t \in [a, b]$. But this contradicts (3) above. Now consider the point $(a^*, 1)$. Since

$$\psi(a^*, 1) = \eta(a^*) = 0,$$

by the above argument, we deduce

$$\frac{\partial \psi}{\partial t}(a^*, 1) \neq 0.$$

Now we consider the set

$$\Sigma = \{(t, \lambda) \in [a, b] \times [0, 1] : \psi(t, \lambda) = 0\}.$$

Now $(a^*, 1) \in \Sigma$ and since $\frac{\partial \psi}{\partial t}(a^*, 1) \neq 0$, we can use the implicit function theorem to deduce the existence of a unique C^1 function $t(\lambda)$ such that $(t(\lambda), \lambda) \in \Sigma$ for all λ in a neighbourhood of 1. Thus, clearly the graph of this function, which describes parametrically a curve γ in the $\lambda - t$ plane, has to enter the open rectangle

$$R = (a, b) \times (0, 1).$$

Now, the curve can not terminate inside R, since if it does, then by virtue of (1.8), the conditions for the implicit function theorem is still satisfied at that point and we can continue the curve. Moreover, differentiationg the expression

$$\psi(t(\lambda), \lambda) = 0,$$

we deduce

$$\frac{d}{d\lambda} t(\lambda) = -\frac{\frac{\partial \psi}{\partial \lambda}}{\frac{\partial \psi}{\partial t}}.$$

This implies, by virtue of (1.8), that $\frac{\partial \psi}{\partial t}$ is finite and thus the curve can not have a 'horizontal' (i.e. paraller to the t-axis) derivative at any point. Thus, the curve can not 'double back' and meet the the line $\lambda = 1$ again. Thus the only other possibilities that remain are the following

Now since $\lambda P + (1 - \lambda) > 0$ for $(t, \lambda) \in [a, b] \times [0, 1)$, we can write this as

$$\ddot{\psi} + \frac{\lambda \dot{P}}{[\lambda P + (1 - \lambda)]} \dot{\psi} = \frac{\lambda}{[\lambda P + (1 - \lambda)]} Q \psi.$$

Clearly, the coefficients are smooth functions of λ, thus the continuous dependence on parameters theorem for solutions of ODEs yield the conclusion about the continuity of the partial derivative with respect to λ.

1. the curve exits R through the line $\lambda = 0$,
2. The curve exits R through the line $t = a$ and
3. the curve exits R through the the line $t = b$.

   Now we show these possibilities are impossible too. The easiest is the first one. This is impossible because for $\lambda = 0$, the functional reduces to $\tilde{I}$ which does not have any conjugate points at all. The curve γ can not meet the line $t = a$ as well. Indeed, since $\psi(a, \lambda) \in \Sigma$ for all $\lambda \in [0, 1]$ and the hypotheses of the implicit function theorem is satisfies on any point (a, λ), the function $t(\lambda)$ satisfying $\psi(t(\lambda), \lambda) = 0$ is unique. But $t(\lambda) \equiv a$ is one such function and clearly γ is not the graph of this function, as γ passes through $(a^*, 1)$. The third possibility is also impossible. If γ intersects the line $t = b$ at the point (b, λ_1) for some $\lambda_1 \in (0, 1)$, then we have

   $$\psi(a, \lambda_1) = 0 = \psi(b, \lambda_1)$$

   with $\psi \not\equiv 0$. But this implies, by Proposition 9 that $J_{\lambda_1}\psi = 0$. But this contradicts the fact that

   $$J_\lambda(\psi) > 0 \qquad \text{for } \lambda \in [0, 1)$$

   for every $\psi \in C^1([a, b])$, $\psi \not\equiv 0$ with $\psi(a) = \psi(b) = 0$.

This proves that no such curve γ can exists and thus there can be no interior conjugate point. $\square$

Remark 14. *Note why the proof does not work for $a^* = b$. As the point $(b, 1)$ is the upper-right corner of the boundary of the rectangle R, it is perfectly possible for a C^1 curve with non-horizontal derivative to pass through $(b, 1)$ and to not enter R ever.*

1.2 Examples

Now we want to show some examples first.

Case 1: Lagrangian depends only on the derivative

$$f(t, u, \xi) = f(\xi).$$

This is the simplest case. The Euler-Lagrange equation is

$$\frac{d}{dt}[f'(\dot{u})] = 0, \quad \text{i.e. } f'(\dot{u}) = \text{constant}.$$

Note that

$$\bar{u}(t) = \frac{\beta - \alpha}{b - a}(t - a) + \alpha \tag{1.9}$$

is a solution of the equation and also satisfies the boundary conditions $\bar{u}(a) = \alpha,\ \bar{u}(b) = \beta$. It is therefore a stationary point of I. It is not, however, always a minimizer of (P) as we shall see.

1. f is convex.

If f is convex, the above $\bar{u}$ is indeed a minimizer. From Jensen inequality, it follows that for any $u \in C^1([a,b])$ with $u(a) = \alpha,\ u(b) = \beta$

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(\dot{u}(t))\ \mathrm{d}t &\geq f\left(\frac{1}{b-a}\int_a^b \dot{u}(t)\ \mathrm{d}t\right) \\
&= f\left(\frac{u(b) - u(a)}{b-a}\right) \\
&= f\left(\frac{\beta-\alpha}{b-a}\right) = f(\dot{\bar{u}}(t)) \\
&= \frac{1}{b-a}\int_a^b f(\dot{\bar{u}}(t))\ \mathrm{d}t
\end{aligned}$$

which is the claim. If f is not strictly convex, then, in general, there are other minimizers.

2. f is non-convex.

If f is non-convex, then (P) has, in general, **no solution** and therefore the above $\bar{u}$ is not necessarily a minimizer (in the particular example below it is a maximizer of the integral).

Consider

$$f(\xi) = e^{-\xi^2}.$$

and

$$(P) \qquad \inf_{u \in X}\left\{I(u) = \int_0^1 f(\dot{u}(t))\,\mathrm{d}t\right\} = m$$

where

$$X = \left\{u \in C^1([0,1]) : u(0) = u(1) = 0\right\}.$$

We have from (1.9) that $\bar{u} \equiv 0$ and it is clearly a maximizer of I in the class of admissible functions X .

However (P) has no minimizer, as we now show. Let us show that $m = 0$. Let $\nu \in \mathbb{N}$ and define

$$u_\nu(x) = \nu\left(x - \frac{1}{2}\right)^2 - \frac{\nu}{4}$$

then $u_\nu \in X$ and

$$I(u_\nu) = \int_0^1 e^{-4\nu^2(x-1/2)^2}\,dx = \frac{1}{2\nu}\int_{-\nu}^{\nu} e^{-y^2}\,dy \to 0 \quad \text{as } \nu \to \infty.$$

Thus $m = 0$, as claimed. But clearly, no function $u \in X$ can satisfy

$$\int_0^1 e^{-(\dot{u}(t))^2} \, \mathrm{d}t = 0$$

and hence (P) has no solution.

Minimizer in $C^1_{\mathbf{piece}}$ are not necessarily C^1

Now we give an example to show that **minimizers in the class $C^1_{\mathbf{piece}}$ might not even be C^1**, thus we can not in general expect a gain of regularity.

Consider

$$f(\xi) = \left(\xi^2 - 1\right)^2 .$$

$$(P_{\text{piece}}) \qquad \inf_{u \in X_{\text{piec}}} \left\{ I(u) = \int_0^1 f(\dot{u}(t)) \, \mathrm{d}t \right\} = m_{\text{piece}}$$

where

$$X_{\text{piece}} = \left\{ u \in C^1_{\text{piec}}([0,1]) : u(0) = u(1) = 0 \right\} .$$

We can easily check that the **tent function**

$$v_1(t) = \begin{cases} t & \text{if } t \in [0, 1/2] \\ 1 - t & \text{if } t \in (1/2, 1] \end{cases}$$

is a minimizer since v is piecewise C^1 and satisfies $v_1(0) = v_1(1) = 0$ and $I(v_1) = 0$. Thus $m_{\text{piece}} = 0$.

Note that (P_{piece}) has a plethora of minimizers, not just one. Indeed, there are uncountably infinitely many minimizers. For example, the **one-sided double tent**

$$v_2(t) = \begin{cases} t & \text{if } t \in [0, 1/4] \\ \dfrac{1}{2} - t & \text{if } t \in [1/4, 1/2] \\ t - \dfrac{1}{2} & \text{if } t \in [1/2, 3/4] \\ 1 - t & \text{if } t \in [3/4, 1] \end{cases}$$

is also a minimizer.

The **two-sided double tent**

$$v_3(t) = \begin{cases} t & \text{if } t \in [0, 1/4] \\ \dfrac{1}{2} - t & \text{if } x \in [1/4, 3/4] \\ t - 1 & \text{if } t \in [3/4, 1] \end{cases}$$

is another one. One can easily construct functions with multiple number of tents, one or two-sided or a combination of those. Any piecewise affine functions with slopes $+1$ or -1 which respects the boundary values is a minimizer. All of them are Lipschitz and of course C^1_{piece}, in fact C^∞_{piece}, but none of them are C^1!

Indeed, the minimization problem in C^1, i.e.

$$(P) \qquad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(\dot{u}(t)) \, \mathrm{d}t \right\} = m$$

where

$$X = \left\{ u \in C^1([0,1]) : u(0) = u(1) = 0 \right\},$$

admits **no** solution.

Let us first show that $m = 0$.

Consider the following sequence, which are just smoothed out versions of v_1 above,

$$u_\nu(t) = \begin{cases} t & \text{if } t \in \left[0, \frac{1}{2} - \frac{1}{\nu}\right] \\ -2\nu^2 \left(t - \frac{1}{2}\right)^3 - 4\nu \left(t - \frac{1}{2}\right)^2 - t + 1 & \text{if } t \in \left(\frac{1}{2} - \frac{1}{\nu}, \frac{1}{2}\right] \\ 1 - t & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Note that $u_\nu \in X$ and

$$I(u_\nu) = \int_0^1 f(\dot{u_\nu}(t)) \, \mathrm{d}t = \int_{\frac{1}{2}-\frac{1}{\nu}}^{\frac{1}{2}} f(\dot{u_\nu}(t)) \, \mathrm{d}t \leq \frac{4}{\nu} \to 0.$$

This implies that indeed $m = 0$. But $I(u) = 0$ implies that $|\dot{u}| = 1$ almost everywhere.

But no function $u \in X$ can satisfy $|\dot{u}| = 1$, since by continuity of the derivative we should have either $\dot{u} = 1$ everywhere or $\dot{u} = -1$ everywhere, which is clearly incompatible with the boundary data.

Also note that the Euler-Lagrange equation is

$$\frac{d}{dt}\left[\dot{u}\left(\dot{u}^2 - 1\right)\right] = 0.$$

It has $\bar{u} \equiv 0$ as a solution. However, since $m = 0$, it is not a minimizer as $I(0) = 1$.

Case 2: Lagrangian depends on time and derivative

$$f(t, u, \xi) = f(t, \xi).$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\left[f_\xi(t, \dot{u})\right] = 0, \quad \text{i.e.} \quad f_\xi(t, \dot{u}) = \text{constant}.$$

The equation is already harder to solve than the preceding one and, in general, it does not have a solution as simple as the last case.

Weierstrass example

Let

$$f(t,\xi) = t\xi^2.$$

Note that $\xi \mapsto f(t,\xi)$ is **convex** for every $t \in [0,1]$ and even **strictly convex** if $t \in (0,1]$. So things would have been very nice without the t-dependence. This example due to Weierstrass is among the first to point out that even t dependence can mess things up.

Consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(t, \dot{u}(t)) \, \mathrm{d}t \right\} = m$$

where

$$X = \left\{ u \in C^1([0,1]) : u(0) = 1, \ u(1) = 0 \right\}.$$

We will show that (P) has no C^1 or piecewise C^1 solution (not even in any Sobolev space).

The Euler-Lagrange equation is

$$\frac{d}{dt}(t\dot{u}) = 0 \quad \Rightarrow \quad \dot{u} = \frac{c}{t} \quad \Rightarrow \quad u(t) = c \log t + d, \ t \in (0,1)$$

where c and d are constants. Observe first that such a u cannot satisfy simultaneously $u(0) = 1$ and $u(1) = 0$.

Let us also consider the following problem

$$(P_{\text{piece}}) \quad \inf_{u \in X_{\text{piece}}} \left\{ I(u) = \int_0^1 f(t, \dot{u}(t)) \, \mathrm{d}t \right\} = m_{\text{piece}}$$

where

$$X_{\text{piece}} = \left\{ u \in C^1_{\text{piece}}([0,1]) : u(0) = 1, \ u(1) = 0 \right\}.$$

We now prove that neither (P) nor (P_{piece}) have a minimizer.

For both cases it is sufficient to establish that $m_{\text{piece}} = m = 0$.

Indeed if there exists a piecewise C^1 function v satisfying $I(v) = 0$, this would imply that $v' = 0$ a.e. in $(0,1)$.

Since the function $v \in X_{\text{piece}}$, it should be continuous and $v(1)$ should be equal to 0. But then this means $v \equiv 0$, which does not verify the other boundary condition, namely $v(0) = 1$. Hence, neither (P) nor (P_{piece}) have a minimizer.

Now let $\nu \in \mathbb{N}$ and consider the sequence

$$u_\nu(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-\log t}{\log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right]. \end{cases}$$

Note that u_ν is piecewise C^1, $u_\nu(0) = 1$, $u_\nu(1) = 0$ and

$$I(u_\nu) = \frac{1}{\log \nu} \to 0 \quad \text{as } \nu \to \infty,$$

hence $m_{\text{piec}} = 0$.

We finally prove that $m = 0$.

Consider the following sequence

$$u_\nu(t) = \begin{cases} \frac{-\nu^2}{\log \nu} t^2 + \frac{\nu}{\log \nu} t + 1 & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-\log t}{\log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right]. \end{cases}$$

We easily have $u_\nu \in X$ and since

$$\dot{u_\nu}(t) = \begin{cases} \frac{\nu}{\log \nu}(1 - 2\nu t) & \text{if } t \in \left[0, \frac{1}{\nu}\right] \\ \frac{-1}{t \log \nu} & \text{if } t \in \left(\frac{1}{\nu}, 1\right] \end{cases}$$

we deduce that

$$0 \le I(u_\nu) = \frac{\nu^2}{\log^2 \nu} \int_0^{1/\nu} t(1 - 2\nu t)^2 \, dt + \frac{1}{\log^2 \nu} \int_{1/\nu}^1 \frac{dt}{t} \to 0, \quad \text{as } \nu \to \infty.$$

This indeed shows that $m = 0$.

Minimizer in C^1 are not necessarily C^2

Our last example shows that even **minimizers in the class C^1 need not automatically have higher regularity, in particular, might not be C^2.**

Consider f which depends on all the variables t, u and ξ, given as

$$f(t, u, \xi) = u^2(2t - \xi)^2.$$

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_{-1}^1 f(t, u(t), \dot{u}(t)) \, dt \right\} = m$$

where $X = \left\{ u \in C^1([0, 1]) : u(-1) = 0, u(1) = 1 \right\}$.

One can easily check that the function

$$v(t) := \begin{cases} 0 & \text{if } t \in [-1, 0] \\ t^2 & \text{if } t \in [0, 1], \end{cases}$$

is a minimizer for (P) which is not C^2.