Introduction to the Calculus of Variations
Lecture Notes
Lecture 21

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Spring 2021, IISc

Chapter 1

Regularity

1.1 Regularity questions in the Calculus of variations

1.2 L^2 regularity

1.2.1 Regularity for harmonic functions

1.2.2 Interior L^2 estimate for elliptic systems

Caccioppoli inequality for elliptic systems

Theorem 1 (Caccioppoli inequality for elliptic systems)**.** *Let $u \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ be a weak solution of*

$$-\operatorname{div}\left(A\left(x\right)\nabla u\right) = f - \operatorname{div} F \qquad \text{in } \Omega,$$

where $f \in L^2\left(\Omega; \mathbb{R}^N\right)$, $F \in L^2\left(\Omega; \mathbb{R}^{N\times n}\right)$ and $A \in L^\infty\left(\Omega; \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n}\right)$. Assume A satisfies the strong Legendre condition, i.e. $\langle A\left(x\right)\xi, \xi\rangle \geq \lambda\left|\xi\right|^2$ for all $\xi \in \mathbb{R}^{N\times n}$ for some $\lambda > 0$. Then for every $x_0 \in \Omega$, $0 < \rho < R < \operatorname{dist}\left(x_0, \partial\Omega\right)$, we have

$$\int_{B_\rho(x_0)} \left|\nabla u\right|^2 \,\mathrm{d}x \leq c\left\{\frac{1}{\left(R-\rho\right)^2}\int_{B_R(x_0)\setminus B_\rho(x_0)} \left|u-\zeta\right|^2 \,\mathrm{d}x \right.$$
$$\left. + R^2\int_{B_R(x_0)} \left|f\right|^2 \,\mathrm{d}x + \int_{B_R(x_0)} \left|F\right|^2 \,\mathrm{d}x\right\}$$

for all $\zeta \in \mathbb{R}^N$, for some constant $c = c\left(\lambda, \left\|A\right\|_{L^\infty}\right) > 0$.

Proof. We first assume $f = 0$. We choose η as before and set $\phi := \left(u-\zeta\right)\eta^2$.

Plugging into the weak formulation, we get using the Legendre condition

$$
\begin{aligned}
&\lambda \int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x \\
&\leq \int_{B_R(x_0)} \eta^2 \left\langle A(x) \nabla u, \nabla u \right\rangle \, \mathrm{d}x \\
&\leq - \int_{B_R(x_0)} \left\langle A(x) \nabla u, 2\eta \nabla \eta \otimes (u - \zeta) \right\rangle \, \mathrm{d}x \\
&\quad + \int_{B_R(x_0)} \eta^2 \left\langle F, \nabla u \right\rangle \, \mathrm{d}x + \int_{B_R(x_0)} \left\langle F, 2\eta \nabla \eta \otimes (u - \zeta) \right\rangle \, \mathrm{d}x \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Now we estimate all three terms. Let $\Lambda = \|A\|_{L^\infty}$. We recall the Young's inequality with $\varepsilon > 0$.

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2.$$

Using Young's inequality with $\varepsilon > 0$ and Young's inequality, we have

$$
\begin{aligned}
I_1 &\leq \varepsilon \Lambda \int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x + \frac{4c^2}{\varepsilon (R - \rho)^2} \int_{B_R(x_0) \setminus B_\rho(x_0)} |u - \zeta|^2 \, \mathrm{d}x \\
I_2 &\leq \varepsilon \int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{B_R(x_0)} |F|^2 \, \mathrm{d}x \\
I_3 &\leq \frac{4c^2}{(R - \rho)^2} \int_{B_R(x_0) \setminus B_\rho(x_0)} |u - \zeta|^2 \, \mathrm{d}x + \int_{B_R(x_0)} |F|^2 \, \mathrm{d}x
\end{aligned}
$$

Choosing $\varepsilon > 0$ small enough, we deduce from the last four inequalities

$$\int_{B_R} |\nabla u|^2 \eta^2 \, \mathrm{d}x \leq c \left\{ \frac{1}{(R - \rho)^2} \int_{B_R \setminus B_\rho} |u - \zeta|^2 \, \mathrm{d}x + \int_{B_R} |F|^2 \, \mathrm{d}x \right\}$$

This gives

$$
\begin{aligned}
\int_{B_\rho} |\nabla u|^2 \, \mathrm{d}x &\leq \int_{B_R} |\nabla u|^2 \eta^2 \, \mathrm{d}x \\
&\leq c \left\{ \frac{1}{(R - \rho)^2} \int_{B_R \setminus B_\rho} |u - \zeta|^2 \, \mathrm{d}x + \int_{B_R} |F|^2 \, \mathrm{d}x \right\}.
\end{aligned}
$$

It remains to prove the theorem when $f \neq 0$. But we can absorb f inside F by writing it as a divergence. This is fairly easy, but we want to keep track of the scaling as well to get the R^2 factor. To this end, define

$$\tilde{f}(y) := R^2 f(Ry + x_0) \qquad \text{for all } y \in B_1(0).$$

Then we find $\tilde{v} \in W_0^{1,2}\left(B_1(0);\mathbb{R}^N\right)$ solving the following problem (e.g. by minimization)

$$\begin{cases} \Delta\tilde{v} = \tilde{f} & \text{in } B_1(0) \\ \tilde{v} = 0 & \text{on } \partial B_1(0). \end{cases}$$

Since $\tilde{v}$ itself can be used as a test function, we obtain using Young's inequality with $\varepsilon > 0$ and Poincaré inequality,

$$\begin{aligned} \int_{B_1(0)} |\nabla\tilde{v}|^2 \, \mathrm{d}x &\leq \int_{B_1(0)} \left|\left\langle \tilde{f}, \tilde{v}\right\rangle\right| \, \mathrm{d}x \\ &\leq \varepsilon \int_{B_1(0)} |\tilde{v}|^2 \, \mathrm{d}x + \frac{1}{\varepsilon}\int_{B_1(0)} \left|\tilde{f}\right|^2 \, \mathrm{d}x \\ &\leq c\varepsilon \int_{B_1(0)} |\nabla\tilde{v}|^2 \, \mathrm{d}x + \frac{1}{\varepsilon}\int_{B_1(0)} \left|\tilde{f}\right|^2 \, \mathrm{d}x \end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we obtain,

$$\int_{B_1(0)} |\nabla\tilde{v}|^2 \, \mathrm{d}x \leq c\int_{B_1(0)} \left|\tilde{f}\right|^2 \, \mathrm{d}x.$$

Now, we set

$$v(x) := \tilde{v}\left(\frac{x - x_0}{R}\right) \qquad \text{for all } x \in B_R(x_0).$$

It is easy to show that $v \in W_0^{1,2}\left(B_R(x_0);\mathbb{R}^N\right)$ and is a weak solution to

$$\operatorname{div}(\nabla v) = \Delta v = f \qquad \text{in } B_R(x_0).$$

Now scaling back to $B_R(x_0)$, we obtain

$$\begin{aligned} \int_{B_R(x_0)} |\nabla v|^2 \, \mathrm{d}x &= R^{n-2}\int_{B_1(0)} |\nabla\tilde{v}|^2 \, \mathrm{d}y \\ &\leq cR^{n-2}\int_{B_1(0)} \left|\tilde{f}\right|^2 \, \mathrm{d}y = R^2\int_{B_R(x_0)} |f|^2 \, \mathrm{d}x. \end{aligned}$$

This completes the proof. □

L^2 regularity

Now we prove the so-called interior $W^{2,2}$ estimate.

Theorem 2 (Interior L^2 estimate). *Let $u \in W^{1,2}\left(\Omega;\mathbb{R}^N\right)$ be a weak solution of the following*

$$-\operatorname{div}(A(x)\nabla u) = f - \operatorname{div} F \qquad \text{in } \Omega,$$

where $f \in L^2\left(\Omega;\mathbb{R}^N\right)$, $F \in W^{1,2}\left(\Omega;\mathbb{R}^{N\times n}\right)$ *and* $A \in W^{1,\infty}\left(\Omega;\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\right)$ *satisfies the strong Legendre condition. Then* $u \in W^{2,2}_{loc}\left(\Omega;\mathbb{R}^N\right)$ *and for any* $\widetilde{\Omega} \subset\subset \Omega$, *we have the estimate*

$$\left\|\nabla^2 u\right\|_{L^2\left(\widetilde{\Omega}\right)} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\nabla F\|_{L^2(\Omega)}\right)$$

where $c > 0$ *is a constant depending only on* $\widetilde{\Omega}$, Ω *and the ellipticity and the bounds on* A.

$W^{2,2}$ regularity for the Laplacian

Before starting the proof, first let us show that since the Laplacian has constant coefficients, the previous Caccioppoli inequality is enough to establish the following special case.

Theorem 3 (Interior L^2 estimate for the Laplacian)**.** *Let* $u \in W^{1,2}\left(\Omega;\mathbb{R}^N\right)$ *be a weak solution of the following*

$$-\Delta u = f \qquad \text{in } \Omega,$$

where $f \in L^2\left(\Omega;\mathbb{R}^N\right)$. *Then* $u \in W^{2,2}_{loc}\left(\Omega;\mathbb{R}^N\right)$ *and for any* $\widetilde{\Omega} \subset\subset \Omega$, *we have the estimate*

$$\left\|\nabla^2 u\right\|_{L^2\left(\widetilde{\Omega}\right)} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right),$$

where $c > 0$ *is a constant depending only on* $\widetilde{\Omega}$ *and* Ω.

Proof. Fix $x_0 \in \Omega$ and $0 < 2R < \operatorname{dist}(x_0, \partial\Omega)$. Let $u_\varepsilon := u * \rho_\varepsilon$ and $f_\varepsilon := f * \rho_\varepsilon$ for some standard symmetric mollifying kernel ρ. Then we can show that

$$-\Delta u_\varepsilon = f_\varepsilon \qquad \text{in } B_{2R}(x_0).$$

Now for any $1 \leq i \leq n$, we deduce

$$-\Delta\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) = \frac{\partial f_\varepsilon}{\partial x_i} \qquad \text{in } B_{2R}(x_0).$$

Thus, writing the weak formulation and integrating by parts, we have for any $\phi \in W^{1,2}_0\left(B_{2R}(x_0);\mathbb{R}^N\right)$,

$$\begin{aligned}\int_{B_{2R}(x_0)} \left\langle \nabla\left(\frac{\partial u_\varepsilon}{\partial x_i}\right), \nabla\phi\right\rangle \,\mathrm{d}x &= \int_{B_{2R}(x_0)} \left\langle \frac{\partial f_\varepsilon}{\partial x_i}, \phi\right\rangle \,\mathrm{d}x \\ &= \int_{B_{2R}(x_0)} \left\langle -f_\varepsilon, \frac{\partial \phi}{\partial x_i}\right\rangle \,\mathrm{d}x \\ &= \int_{B_{2R}(x_0)} \left\langle F, \nabla\phi\right\rangle \,\mathrm{d}x,\end{aligned}$$

where $F := (F_1, \ldots, F_n)$ with $F_i = -f_\varepsilon$ and $F_j = 0$ for $1 \leq j \leq n$ with $j \neq i$. Note that this is the weak formulation of

$$-\Delta \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) = -\operatorname{div} F \qquad \text{in } B_{2R}(x_0) .$$

So applying the Caccioppoli inequality with $\zeta = 0$, we have,

$$\begin{aligned} \int_{B_{R/4}} \left| \nabla \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right|^2 \, \mathrm{d}x &\leq c \left(\frac{1}{R^2} \int_{B_{R/2} \setminus B_{R/4}} \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^2 \, \mathrm{d}x + \int_{B_{R/2}} |F|^2 \, \mathrm{d}x \right) \\ &\leq c \left(\frac{1}{R^2} \int_{B_{R/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{R/2}} |f_\varepsilon|^2 \, \mathrm{d}x \right) . \end{aligned}$$

But now since

$$-\Delta u_\varepsilon = f_\varepsilon \qquad \text{in } B_{2R}(x_0) ,$$

applying Caccioppoli once again, we deduce

$$\int_{B_{R/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x \leq c \left(\frac{1}{R^2} \int_{B_R} |u_\varepsilon|^2 \, \mathrm{d}x + R^2 \int_{B_R} |f_\varepsilon|^2 \, \mathrm{d}x \right) .$$

Combining, we get

$$\int_{B_{R/4}} \left| \nabla \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right|^2 \, \mathrm{d}x \leq c \left(\frac{1}{R^4} \int_{B_R} |u_\varepsilon|^2 \, \mathrm{d}x + \left(R^2 + 1 \right) \int_{B_R} |f_\varepsilon|^2 \, \mathrm{d}x \right) .$$

Choosing $R > 0$ small enough , we get

$$\int_{B_{R/4}} \left| \nabla \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right|^2 \, \mathrm{d}x \leq \frac{c}{R^4} \left(\int_{B_R} |u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_R} |f_\varepsilon|^2 \, \mathrm{d}x \right) .$$

Since this is true for any $1 \leq i \leq n$, we deduce

$$\int_{B_{R/4}} \left| \nabla^2 u_\varepsilon \right|^2 \, \mathrm{d}x \leq \frac{c}{R^4} \left(\int_{B_R} |u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_R} |f_\varepsilon|^2 \, \mathrm{d}x \right) .$$

Since $u_\varepsilon \to u$ and $f_\varepsilon \to f$, we deduce that $\left\| \nabla^2 u_\varepsilon \right\|_{L^2\left(B_{R/4}(x_0)\right)}$ is uniformly bounded and hence weakly convergent. But the weak limit can only be $\nabla^2 u$. Now by passing to the limit $\varepsilon \to 0$, we get

$$\begin{aligned} \int_{B_{R/4}} \left| \nabla^2 u \right|^2 \, \mathrm{d}x &\leq \liminf_{\varepsilon \to 0} \int_{B_{R/4}} \left| \nabla^2 u_\varepsilon \right|^2 \, \mathrm{d}x \\ &\leq \frac{c}{R^4} \left(\int_{B_R} |u|^2 \, \mathrm{d}x + \int_{B_R} |f|^2 \, \mathrm{d}x \right) . \end{aligned}$$

This completes the proof by by a covering argument. $\square$

Remark 4. *Note that the constant blows up as $R \to 0$, so we really need $\widetilde{\Omega} \subset\subset \Omega$ for the covering arguement to work. Also, this is how the constant depends on $\widetilde{\Omega}$ and Ω.*

Nirenberg's difference quotient method

Now we attempt the general case. The trouble here is that since the operator does not have constant coefficients, we can not claim that the derivatives of u satisfies the same type of equation. So instead we work with difference quotients and use the properties of difference quotients we proved in Lecture 12 (Characterization of difference quotients theorem).

Proof. We need to prove just the local estimate on balls. More precisely, for any $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, we need to show

$$\int_{B_{R/4}(x_0)} \left|\nabla^2 u\right|^2 \, \mathrm{d}x \leq c \left(\int_{B_R(x_0)} |u|^2 \, \mathrm{d}x + \int_{B_R(x_0)} |f|^2 \, \mathrm{d}x + \int_{B_R(x_0)} |\nabla F|^2 \, \mathrm{d}x \right),$$

where the constant $c > 0$ can depend on R, λ and $\|A\|_{W^{1,\infty}}$. The result follows from this this by a covering argument. Writing f as a divergence (but this time using the $W^{2,2}$ estimate for the Laplacian), it is enough to prove for the case $f = 0$.

With $f = 0$, the weak formulation becomes

$$\int_\Omega \langle A(x) \nabla u(x), \nabla \phi(x) \rangle \, \mathrm{d}x = \int_\Omega \langle F(x), \nabla \phi(x) \rangle \, \mathrm{d}x$$

for any $\phi \in C_c^\infty\left(\Omega; \mathbb{R}^N\right)$. For any $1 \leq i \leq n$ and for $h \in \mathbb{R}$ with $|h|$ small, we can plugg $\phi(x - he_i)$ as the test function and after a change of variables, we obtain

$$\int_\Omega \langle A(x + he_i) \nabla u(x + he_i), \nabla \phi(x) \rangle \, \mathrm{d}x = \int_\Omega \langle F(x + he_i), \nabla \phi(x) \rangle \, \mathrm{d}x$$

Subtracting the previous identity from this one and diving by h, we obtain

$$\begin{aligned} \int_\Omega \langle A(x + he_i) D_{h,i}(\nabla u), \nabla \phi \rangle \, \mathrm{d}x + \int_\Omega \langle (D_{h,i} A) \nabla u, \nabla \phi \rangle \, \mathrm{d}x \\ = \int_\Omega \langle (D_{h,i} F), \nabla \phi \rangle \, \mathrm{d}x \end{aligned}$$

Note that

$$D_{h,i}(\nabla u) = \nabla (D_{h,i} u)$$

Thus, we get

$$\begin{aligned} \int_\Omega \langle A(x + he_i) \nabla (D_{h,i} u), \nabla \phi \rangle \, \mathrm{d}x + \int_\Omega \langle (D_{h,i} A) \nabla u, \nabla \phi \rangle \, \mathrm{d}x \\ = \int_\Omega \langle (D_{h,i} F), \nabla \phi \rangle \, \mathrm{d}x \end{aligned}$$

Applying the Caccioppoli inequality, we deduce, for any $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$,

$$\int_{B_{R/4}(x_0)} |\nabla (D_{h,i}u)|^2 \, \mathrm{d}x \leq \frac{c}{R^2} \int_{B_{R/2}(x_0)} |D_{h,i}u|^2 \, \mathrm{d}x + c\int_{B_{R/2}(x_0)} |D_{h,i}A|^2 |\nabla u|^2 \, \mathrm{d}x + c\int_{B_{R/2}(x_0)} |D_{h,i}F|^2 \, \mathrm{d}x$$

Since u and F are both $W^{1,2}$ and A is $W^{1,\infty}$, the RHS stays uniformly bounded as $h \to 0$. Thus,

$$\int_{B_{R/4}(x_0)} |D_{h,i}(\nabla u)|^2 \, \mathrm{d}x = \int_{B_{R/4}(x_0)} |\nabla (D_{h,i}u)|^2 \, \mathrm{d}x$$

stays uniformly bounded as $h \to 0$.

This implies

$$\int_{B_{R/4}(x_0)} \left| \frac{\partial}{\partial x_i} (\nabla u) \right|^2 \, \mathrm{d}x < \infty$$

and letting $h \to 0$, we obtain

$$\begin{aligned} \int_{B_{R/4}(x_0)} \left| \frac{\partial}{\partial x_i} (\nabla u) \right|^2 \, \mathrm{d}x &\leq \frac{c}{R^2} \int_{B_{R/2}(x_0)} |\nabla u|^2 \, \mathrm{d}x + c \|\nabla A\|_{L^\infty} \int_{B_{R/2}(x_0)} |\nabla u|^2 \, \mathrm{d}x \\ &\quad + c\int_{B_{R/2}(x_0)} |\nabla F|^2 \, \mathrm{d}x. \end{aligned}$$

But since this is true for any $1 \leq i \leq n$, we have

$$\int_{B_{R/4}(x_0)} \left|\nabla^2 u\right|^2 \, \mathrm{d}x \leq c\left(R, \|A\|_{W^{1,\infty}}\right) \int_{B_{R/2}(x_0)} |\nabla u|^2 \, \mathrm{d}x + c\int_{B_{R/2}(x_0)} |\nabla F|^2 \, \mathrm{d}x.$$

Applying Caccioppoli inequality once again to estimate the gradient term, we obtain

$$\int_{B_{R/4}(x_0)} \left|\nabla^2 u\right|^2 \, \mathrm{d}x \leq c\left(\int_{B_R(x_0)} |u|^2 \, \mathrm{d}x + c\int_{B_R(x_0)} |\nabla F|^2 \, \mathrm{d}x \right),$$

where the constant $c > 0$ this time depends on R, λ, and $\|A\|_{W^{1,\infty}}$. This completes the proof. $\square$

Now we are going to show another interesting corollary of the Caccioppoli inequality.

1.2.3 Hole filling technique

Now we are going to prove a decay estimate for for the gradient. The method is known as the 'hole filling technique' of Widman.

Proposition 5. *Let $u \in W^{1,2}\left(\Omega;\mathbb{R}^N\right)$ be a weak solution of*

$$-\operatorname{div}\left(A\left(x\right)\nabla u\right)=0 \qquad \text{in } \Omega,$$

where $A \in L^{\infty}\left(\Omega;\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\right)$. Assume A satisfies the strong Legendre condition. Then there exists an $\alpha=\alpha\left(\lambda,\|A\|_{L^{\infty}}\right)>0$ such that for every $x_0\in\Omega$, $0<\rho<\operatorname{dist}\left(x_0,\partial\Omega\right)$, we have

$$\int_{B_{\rho}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq c\rho^{\alpha}.$$

Proof. For every $x_0\in\Omega$, $0<R<\operatorname{dist}\left(x_0,\partial\Omega\right)$, applying the Caccioppoli inequality, we get

$$\int_{B_{R/2}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq c\frac{1}{R^2}\int_{B_R(x_0)\setminus B_{R/2}(x_0)}\left|u-(u)_{B_R\setminus B_{R/2}}\right|^2\,\mathrm{d}x$$

Applying Poincaré inequality, this implies

$$\int_{B_{R/2}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq c\int_{B_R(x_0)\setminus B_{R/2}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x.$$

Filling the hole, we obtain,

$$\int_{B_{R/2}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq\left(\frac{c}{c+1}\right)\int_{B_R(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x.$$

Iterating, we have

$$\int_{B_{R/2^k}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq\left(\frac{c}{c+1}\right)^k\int_{B_R(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x.$$

Since $\frac{c}{c+1}<1$, the last one is a decay estimate. Then for any $0<\rho<R$, we have by interpolating,

$$\int_{B_{\rho}(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x\leq 2^{\alpha}\left(\frac{\rho}{R}\right)^{\alpha}\int_{B_R(x_0)}\left|\nabla u\right|^2\,\mathrm{d}x,$$

where $\alpha:=\log_2\left(\frac{c+1}{c}\right)$. This proves the result. □

This immediately implies that if u is a weak solution of

$$-\operatorname{div}\left(A\left(x\right)\nabla u\right)=0 \qquad \text{in } \mathbb{R}^n$$

with **finite energy**, then u is constant. We also have the

Theorem 6 (Liouville theorem)**.** *Let $u \in W^{1,2}_{loc}\left(\mathbb{R}^2;\mathbb{R}^N\right)$ be a weak solution of*

$$-\operatorname{div}\left(A\left(x\right)\nabla u\right) = 0 \qquad \text{in } \mathbb{R}^2,$$

where $A \in L^\infty\left(\Omega;\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\right)$ and satisfies the strong Legendre condition. If u is L^∞, then u is constant.

Proof. By Caccioppoli inequality, we have for any $R > 0$, we get

$$\int_{B_R(0)} |\nabla u|^2 \ \mathrm{d}x \leq c\frac{1}{R^2}\int_{B_{2R}(0)} |u|^2 \ \mathrm{d}x \leq c\sup_{\mathbb{R}^2}|u|^2 .$$

Hence, by the inequality we derived in the proof of last result, we have for any $0 < \rho < R$,

$$\int_{B_\rho(x_0)} |\nabla u|^2 \ \mathrm{d}x \leq 2^\alpha\left(\frac{\rho}{R}\right)^\alpha\int_{B_R(x_0)} |\nabla u|^2 \ \mathrm{d}x \leq c2^\alpha\left(\frac{\rho}{R}\right)^\alpha\sup_{\mathbb{R}^2}|u|^2 .$$

Letting $R \to \infty$, we obtain the conclusion. $\square$