Introduction to the Calculus of Variations
Lecture Notes
Lecture 20

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Chapter 1

Direct methods

1.1 Dirichlet Integral

1.2 Integrands depending only on the gradient

1.3 Integrands with x dependence

1.4 Integrands with x and u dependence

1.5 Euler-Lagrange Equations

1.6 Glimpses of the Vectorial Calculus of Variations

1.6.1 The determinant

Quasiconvexity of the determinant

Proposition 1. *Let $n = N = 2$. Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as*

$$f(\xi) = \det \xi.$$

Then f is quasiaffine.

Proof. For any $\xi \in \mathbb{R}^{2\times 2}$, for any bounded open set $D \subset \mathbb{R}^2$ and any $\phi \in$

$W_0^{1,\infty}\left(D;\mathbb{R}^2\right)$, we have,

$$\begin{aligned}
&\det\left(\xi+\nabla\phi\right)\\
&\quad=\det\begin{pmatrix}\xi_{11}+\dfrac{\partial\phi^1}{\partial x_1} & \xi_{12}+\dfrac{\partial\phi^1}{\partial x_2}\\ \xi_{21}+\dfrac{\partial\phi^2}{\partial x_1} & \xi_{22}+\dfrac{\partial\phi^2}{\partial x_2}\end{pmatrix}\\
&\quad=\left(\xi_{11}\xi_{22}-\xi_{12}\xi_{21}\right)+\left(\xi_{11}\frac{\partial\phi^2}{\partial x_2}+\xi_{22}\frac{\partial\phi^1}{\partial x_1}-\xi_{12}\frac{\partial\phi^2}{\partial x_1}-\xi_{21}\frac{\partial\phi^1}{\partial x_2}\right)\\
&\quad=\det\xi+\left(\xi_{11}\frac{\partial\phi^2}{\partial x_2}+\xi_{22}\frac{\partial\phi^1}{\partial x_1}-\xi_{12}\frac{\partial\phi^2}{\partial x_1}-\xi_{21}\frac{\partial\phi^1}{\partial x_2}\right).
\end{aligned}$$

Note that since ϕ has zero trace on ∂D, integrating by parts, we have

$$\int_D \xi_{11}\frac{\partial\phi^2}{\partial x_2}\left(y\right)\ \mathrm{d}y=0.$$

Similarly,

$$\int_D \xi_{22}\frac{\partial\phi^1}{\partial x_1}\left(y\right)\ \mathrm{d}y,\int_D \xi_{12}\frac{\partial\phi^2}{\partial x_1}\left(y\right)\ \mathrm{d}y,\int_D \xi_{21}\frac{\partial\phi^1}{\partial x_2}\left(y\right)\ \mathrm{d}y=0.$$

So, we deduce from the earlier computation,

$$\int_D \det\left(\xi+\nabla\phi\left(y\right)\right)\ \mathrm{d}y=\int_D\det\xi\ \mathrm{d}y=\left|D\right|\det\xi.$$

This completes the proof. □

Thus the determinant is both quasiaffine and rank one affine, but neither affine nor convex. We now introduce another notion of convexity in the vectorial calculus of variations.

1.6.2 Polyconvexity

Definition 2 (Polyconvexity for $n=N=2$)**.** *A function $f:\mathbb{R}^{2\times 2}\to\mathbb{R}$ is called **polyconvex** if there exists a convex function $F:\mathbb{R}^{2\times 2}\times\mathbb{R}\to\mathbb{R}$ such that*

$$f\left(\xi\right)=F\left(\xi,\det\xi\right)\qquad\textit{for all } \xi\in\mathbb{R}^{2\times 2}.$$

Remark 3. *This is not the general definition of polyconvexity. This is what the general definition reduces to in the case $n=N=2$.*

We have already proved the weak continuity of determinants. Using that result and the Mazur lemma, we can prove the weak lower semicontinuity of functionals with polyconvex integrands.

1.6.3 Weak lower semicontinuity for polyconvex integrands

Theorem 4 (wlsc for polyconvex integrands)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be open, bounded and smooth and let* $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be convex and continuous. Let*

$$I[u] := \int_\Omega F(\nabla u(x), \det \nabla u(x)) \, \mathrm{d}x.$$

Let

$$\begin{cases} u_s \rightharpoonup u & \text{in } W^{1,2}\left(\Omega; \mathbb{R}^2\right), \\ \det \nabla u_s \rightharpoonup \det \nabla u & \text{in } L^1(\Omega). \end{cases}$$

Then, $\quad \liminf_{s\to\infty} I[u_s] \geq I[u]$.

The integrand need **not be convex** as a function of the gradient variable. Also, if $u_s \rightharpoonup u$ in $W^{1,p}$ for some $2 < p < \infty$, both the convergences in the hypothesis above are satisfied and consequently the theorem holds.

1.6.4 Existence for polyconvex integrands

Theorem 5 (existence for polyconvex integrands)**.** *Let* $2 \leq p < \infty$ *and let* $\Omega \subset \mathbb{R}^2$ *be open bounded and smooth. Let* $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^2\right)$ *be given. Let* $F : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $F = F(\xi, \theta)$ *be continuous, convex and satisfies,*

$$f(\xi, \theta) \geq \begin{cases} c_1 |\xi|^2 + c_2 |\theta|^q & \text{if } p = 2, \\ c_1 |\xi|^p & \text{if } p > 2, \end{cases} \qquad \text{for all } \xi \in \mathbb{R}^{2\times 2}, \theta \in \mathbb{R},$$

for some $c_1, c_2 > 0$, *some exponent* $q > 1$. *Let*

$$I[u] := \int_\Omega F(\nabla u(x), \det \nabla u(x)) \, \mathrm{d}x.$$

If $I[u_0] < \infty$, *then the following problem*

$$\inf \left\{ I[u] : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^2\right) \right\} = m$$

admits a minimizer.

Proof. We only show the case $p = 2$. The other case is much easier. For any minimizing sequence $\{u_s\}_{s\geq 1}$, the coercivity inequality implies

$$\int_\Omega |\nabla u_s(x)|^2 \, \mathrm{d}x \leq \frac{1}{c_1}(m+1)$$

and

$$\int_\Omega |\det \nabla u_s(x)|^q \, \mathrm{d}x \leq \frac{1}{c_2}(m+1)$$

for all $s \geq 1$. By Poincaré inequality, the first estimate implies that $\{u_s\}_{s\geq 1}$ is uniformly bounded in $W^{1,2}\left(\Omega;\mathbb{R}^2\right)$. Hence, up to the extraction of a subsequence, we have

$$u_s \rightharpoonup u \qquad \text{in } W^{1,2}\left(\Omega;\mathbb{R}^2\right),$$

for some $u \in u_0 + W_0^{1,2}\left(\Omega;\mathbb{R}^2\right)$.

The second inequality implies that $\{\det \nabla u_s\}_{s\geq 1}$ is uniformly bounded in $L^q(\Omega)$ and since $q > 1$, up to the extraction of a subsequence, we have

$$\det \nabla u_s \rightharpoonup v \qquad \text{in } L^q(\Omega),$$

for some $v \in L^q(\Omega)$. But using the same argument as in the proof of weak continuity of the determinant result, by uniqueness of limits, we must have

$$v = \det \nabla u.$$

Thus, we have

$$\begin{cases} u_s \rightharpoonup u & \text{in } W^{1,2}\left(\Omega;\mathbb{R}^2\right), \\ \det \nabla u_s \rightharpoonup \det \nabla u & \text{in } L^1(\Omega). \end{cases}$$

Now we can use the wlsc theorem to conclude. □

Chapter 2

Regularity

2.1 Regularity questions in the Calculus of variations

Now we begin studying the question of regularity of minimizers. We have established the existence of a minimizer in some Sobolev class, typically $W^{1,p}$. Now we want to show that they are in fact more regular when the problem allows it. This is in general a quite difficult subject which is both important, interesting and intricate. The results can be broadly divided into two types, depending on the regularity of the integrand.

- Regular enough integrands: We establish regularity for the Euler-Lagrange equations.
- Integrands without the required regularity: Here we can no longer work with the Euler-Lagrange equation and instead prove regularity directly using minimality.

The techniques used for both types are related, but the latter is usually considerably more technically challenging. In this course, we would only discuss regularity results for the Euler-Lagrange equations. Since the Euler-Lagrange equations are often 'elliptic', these types of results are called **elliptic regularity** results.

2.1.1 Panorama of different types of Elliptic regularity results

Motivation

To appreciate the rather remarkable nature of the results we are going to prove, let us start with a simple question.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. Suppose we know $\Delta u \in L^2(\Omega)$ for some $u \in W^{1,2}(\Omega)$. Since the Laplacian is a polynomial of the second derivatives of u, can we say something about the second derivatives of u?

First consider the case $n = 1$. Here the statement $\Delta u \in L^2$ reduces to $\ddot{u} \in L^2$. This, together with the fact that $u \in W^{1,2}$ immediately implies $u \in W^{2,2}$, at least locally inside Ω. Can we say the same for any $n \geq 1$?

Things are far from clear when $n \geq 2$. For example, if $n = 2$,

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y).$$

So first of all, all second derivatives of u does not even appear in the equation, only the pure ones do.

Secondly, it is not even clear if we can conclude that the pure second derivatives are in L^2.

We know only their sum to be L^2 to begin with.

It is perfectly possible for the sum of two functions, none of which are L^2, to be square integrable.

However, this somewhat miraculous conclusion is actually true in all dimensions.

Types of Elliptic regularity results

Elliptic regularity results can be broadly classified into a few types depending on the techniques and the spaces involved.

- **Linear/Perturbative theory** Here we first establish regularity for a model constant coefficient operator and then tackle the variable coefficient operator case by perturbation techniques. Depending on the spaces involved they can be classified into three types.

  - L^2 theory: This implies results of the type

    $$P(x,D)u \in L^2 \Rightarrow u \in W^{2,2}$$

  - L^p theory: This implies results of the type

    $$P(x,D)u \in L^p \Rightarrow u \in W^{2,p}$$

  - Schauder theory: This implies results of the type

    $$P(x,D)u \in C^{0,\alpha} \Rightarrow u \in C^{2,\alpha}$$

  Here $P(x,D)$ is a second order variable coefficient linear operator with appropriately regular coefficients. All of these has their local and up to the boundary versions.

- **Non Linear theory** Here we first establish regularity for a model linear operator, but with rough coefficients so that Perturbative theory can not be applied. This can then be used to conclude regularity for nonlinear problems as well. These can be divided into two classes.
  - **Equations:** Typically, these results are called **De Giorgi-Nash-Moser** theory. The prototype result is

    $$P(x, D)\, u = 0 \Rightarrow u \in C^{1,\alpha}_{loc}$$

    These results are typically valid for equations ($N = 1$) and does not in general extend to systems ($N \geq 2$), except for systems with special structures.
  - **General systems:** Everywhere regularity, is in general **not true** for nonlinear elliptic systems. Instead, we try to prove what is known as **partial regularity** results. The prototype result is

    $$P(x, D)\, u = 0 \text{ in } \Omega \Rightarrow u \in C^{1,\alpha}_{loc}(\Omega \setminus \Sigma)$$

    where Σ is a 'lower dimensional set', called the singular set.

2.2 L^2 regularity

2.2.1 Interior L^2 regularity

We can barely scratch the surface of elliptic regularity in this course. That would require a course for itself. We would only prove the so-called interior $W^{2,2}$ estimate.

Theorem 6 (Interior L^2 estimate)**.** *Let $u \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ be a weak solution of the following*

$$-\operatorname{div}\left(A(x)\nabla u\right) = f - \operatorname{div} F \qquad \textit{in } \Omega,$$

where $f \in L^2\left(\Omega; \mathbb{R}^N\right)$, $F \in W^{1,2}\left(\Omega; \mathbb{R}^{N\times n}\right)$ and $A \in W^{1,\infty}\left(\Omega; \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n}\right)$ satisfies the strong Legendre condition. Then $u \in W^{2,2}_{loc}\left(\Omega; \mathbb{R}^N\right)$ and for any $\widetilde{\Omega} \subset\subset \Omega$, we have the estimate

$$\left\|\nabla^2 u\right\|_{L^2(\widetilde{\Omega})} \leq c\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\nabla F\|_{L^2(\Omega)}\right)$$

where $c > 0$ is a constant depending only on $\widetilde{\Omega}$, Ω and the ellipticity and the bounds on A.

2.2.2 Regularity for Harmonic functions

Caccioppoli inequality for harmonic functions

The main tool is an inequality called the **Caccioppoli inequality** or the **reverse Poincaré inequality**. We begin with the simplest case, the case of harmonic functions.

Theorem 7 (Caccioppoli inequality). *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = 0$, i.e.*

$$\int_{\Omega} \langle \nabla u, \nabla \phi \rangle = 0 \qquad \text{for every } \phi \in W_0^{1,2}(\Omega). \tag{2.1}$$

Then for every $x_0 \in \Omega$, $0 < \rho < R < \operatorname{dist}(x_0, \partial\Omega)$, we have

$$\int_{B_\rho(x_0)} |\nabla u|^2 \, \mathrm{d}x \leq \frac{c}{(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\lambda|^2 \, \mathrm{d}x, \qquad \text{for all } \lambda \in \mathbb{R}, \tag{2.2}$$

for some universal constant $c > 0$.

The regularity is a consequence of the competition between reverse Poincaré and the usual Poincaré-Sobolev inequalities.

Proof. Let $\eta \in C_c^\infty(B_R(x_0))$ be such that

$$\eta \equiv 1 \quad \text{in } B_\rho(x_0), \quad 0 \leq \eta \leq 1 \quad \text{and} \quad |\nabla \eta| \leq \frac{c}{R-\rho}.$$

For any $\lambda \in \mathbb{R}$, set $\phi = (u-\lambda)\eta^2$. Plugging into (2.1), we get

$$\int_{\Omega} |\nabla u|^2 \eta^2 \, \mathrm{d}x + \int_{\Omega} \langle \nabla u, (u-\lambda) 2\eta \nabla \eta \rangle \, \mathrm{d}x = 0.$$

Thus, using Hölder inequality, we deduce

$$\begin{aligned} \int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x &\leq \int_{B_R(x_0)} |\nabla u| \, |u-\lambda| \, 2\eta \, |\nabla \eta| \, \mathrm{d}x \\ &\leq \left(\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_R(x_0)} 4 |u-\lambda|^2 |\nabla \eta|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \end{aligned}$$

This implies

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x \leq 4 \int_{B_R(x_0)} |u-\lambda|^2 |\nabla \eta|^2 \, \mathrm{d}x.$$

Hence, we obtain

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla u|^2 \, \mathrm{d}x &\leq \int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, \mathrm{d}x \\ &\leq 4 \int_{B_R(x_0)} |u-\lambda|^2 |\nabla \eta|^2 \, \mathrm{d}x \\ &\leq \frac{4c^2}{(R-\rho)^2} \int_{B_R(x_0)\setminus B_\rho(x_0)} |u-\lambda|^2 \, \mathrm{d}x.\end{aligned}$$

This completes the proof. □

However remarkable it may sound, this is enough to proof that harmonic functions are smooth! To do this, we would use what are called **apriori estimates**. This is a baffling notion at first sight. To prove the smoothness of u, first we are going to prove some estimates assuming u is smooth!! In case you are wondering, we do know how to spell circularity.

Apriori estimates for higher derivatives

Proposition 8 (Local apriori estimates for higher derivatives)**.** *Let $u \in C^\infty(\Omega)$ be a **smooth** solution of $\Delta u = 0$. Then for every $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$ and any $k \in \mathbb{N}$, we have*

$$\int_{B_{R/2}(x_0)} \left|D^k u\right|^2 \, \mathrm{d}x \leq c \int_{B_R(x_0)} |u|^2 \, \mathrm{d}x$$

for some constant $c = c(k, R) > 0$.

Proof. Since u is harmonic and smooth, so is $\frac{\partial u}{\partial x_i}$ for any $1 \leq i \leq n$. So applying the Caccioppoli inequality, for any $x_0 \in \Omega$ and $0 < R < \operatorname{dist}(x_0, \partial\Omega)$, we have, for any $1 \leq i \leq n$,

$$\int_{B_{R/2}(x_0)} \left|\nabla\left(\frac{\partial u}{\partial x_i}\right)\right|^2 \, \mathrm{d}x \leq \frac{c}{R^2} \int_{B_{2R/3}(x_0)} \left|\frac{\partial u}{\partial x_i}\right|^2 \, \mathrm{d}x \leq \frac{c}{R^4} \int_{B_R(x_0)} |u|^2 \, \mathrm{d}x.$$

We can iterate for higher derivatives. □

Local smoothness of harmonic functions

Theorem 9 (Smoothness of harmonic functions)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\Delta u = 0$, i.e.*

$$\int_\Omega \langle \nabla u, \nabla \phi \rangle = 0 \qquad \textit{for every } \phi \in W_0^{1,2}(\Omega).$$

Then $u \in C^\infty_{loc}(\Omega)$ and for every $x_0 \in \Omega$, $0 < R < \operatorname{dist}(x_0, \partial\Omega)$ and any $k \in \mathbb{N}$, we have

$$\int_{B_{R/2}(x_0)} \left|D^k u\right|^2 \, \mathrm{d}x \leq c \int_{B_R(x_0)} |u|^2 \, \mathrm{d}x$$

for some constant $c = c(k, R) > 0$.

Proof. Fix $x_0 \in \Omega$ and $0 < R < \operatorname{dist}(x_0, \partial\Omega)$. Let $u_\varepsilon := u * \rho_\varepsilon$, for some standard symmetric mollifying kernel ρ. Then using Fubini, we can show that u_ε is harmonic in a neighbouhood of $B_R(x_0) \subset \Omega$. Thus, any derivative of u_ε of any order satisfies the apriori estimates and thus $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $W^{k,2}\left(B_{R/2}(x_0)\right)$ for any $k \in \mathbb{N}$. By Rellich-Kondrachov compact embeddings, this implies that up to the extraction of a subsequence,

$$u_\varepsilon \to u \qquad \text{in } C^m\left(\overline{B_{R/2}(x_0)}\right)$$

for any $m \in \mathbb{N}$. Since x_0 and R are otherwise arbitrary, this proves $u \in C^\infty_{loc}(\Omega)$. The estimates now follows from the estimates for u_ε by passing to the limit. $\square$