Introduction to the Calculus of Variations
Lecture Notes
Lecture 19

Swarnendu Sil

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Chapter 1

Direct methods

1.1 Dirichlet Integral

1.2 Integrands depending only on the gradient

1.3 Integrands with x dependence

1.4 Integrands with x and u dependence

1.5 Euler-Lagrange Equations

1.5.1 Growth conditions

Now we want to derive the Euler-Lagrange equation satisfied by a minimizer. But this would require certain regularity of the integrand f. So far, we have only worked with the assumption that f is a Carathéodory function satisfying some coercivity conditions. Now we need to assume something more, which are called growth conditions. These tells us how $|f(x,u,\xi)|$ grows when $|u|, |\xi| \to \infty$.

Growth conditions on f

Definition 1 (Growth condition on f)**.** *Let $1 < p < \infty$. A Carathéodory function*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \qquad f = f(x,u,\xi)$$

*is said to satisfy p-**growth conditions** if there exists $\alpha \in L^1(\Omega)$ and $\beta \geq 0$ such that*

$$|f(x,u,\xi)| \leq \alpha(x) + \beta\left(|u|^p + |\xi|^p\right) \tag{G_p}$$

for a.e. $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

Note that the p-growth conditions automatically implies that

$$I[u] := \int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}x < \infty$$

for every $u \in W^{1,p}(\Omega; \mathbb{R}^N)$.

Growth conditions on the derivatives of f

Now we need some growth conditions on the derivatives of f. There are a few growth conditions that are used in practice. We only state here the perhaps the most used one.

Definition 2 (Controllable p-growth conditions)**.** *Let $1 < p < \infty$. A Carathéodory function $f = f(x, u, \xi)$ is said to satisfy **controllable p-growth conditions** if f_{u^i} and $f_{\xi^i_\alpha}$ are Carathéodory functions for every $1 \leq i \leq N$ and $1 \leq \alpha \leq n$ and these functions satisfy the estimates*

$$\left.\begin{aligned} |D_u f(x, u, \xi)| &\leq \alpha_1(x) + \beta\left(|u|^{p-1} + |\xi|^{p-1}\right) \\ |D_\xi f(x, u, \xi)| &\leq \alpha_2(x) + \beta\left(|u|^{p-1} + |\xi|^{p-1}\right) \end{aligned}\right\} \qquad (G_{p,\mathrm{cont}})$$

for a.e. $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ for some $\alpha_1, \alpha_2 \in L^1(\Omega)$ and $\beta \geq 0$

1.5.2 Euler-Lagrange equations

Theorem 3 (Euler-Lagrange equations)**.** *Let $n \geq 2, N \geq 1$ be integers, $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and $1 < p < \infty$. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(x, u, \xi)$ satisfy (G_p) and $(G_{p,\mathrm{cont}})$. Suppose $\bar{u} \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$ is a minimizer for*

$$\inf\left\{ I[u] : u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N) \right\} = m$$

Then for every $\phi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$, we have

$$\int_\Omega \left[\langle D_\xi f(x, \bar{u}, \nabla\bar{u}), \nabla\phi\rangle + \langle D_u f(x, \bar{u}, \nabla\bar{u}), \phi\rangle \right] \, \mathrm{d}x = 0.$$

In other words, $\bar{u}$ is a 'weak' solution for the Dirichlet BVP for the (system of) PDE

$$\begin{cases} \operatorname{div}\left[D_\xi f(x, u, \nabla u)\right] = D_u f(x, u, \nabla u) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Proof. By (G_p), we have $I[\bar{u} + \varepsilon\phi]$ is well defined for every $\varepsilon \in \mathbb{R}$ and every $\phi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$. Since $\bar{u}$ is a minimizer, we must have

$$0 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(I[\bar{u} + \varepsilon\phi] - I[\bar{u}] \right)$$

Now we compute

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left(I\left[\bar{u}+\varepsilon\phi\right]-I\left[\bar{u}\right]\right) \\
&\quad=\frac{1}{\varepsilon}\int_{\Omega}\mathrm{d}x\int_{0}^{1}\frac{d}{dt}\left[f\left(x,\bar{u}\left(x\right)+t\varepsilon\phi\left(x\right),\nabla\bar{u}\left(x\right)+t\varepsilon\nabla\phi\left(x\right)\right)\right]\mathrm{d}t \\
&\quad=\int_{\Omega}g\left(x,\varepsilon\right)\mathrm{d}x,
\end{aligned}
$$

where

$$
g\left(x,\varepsilon\right):=\int_{0}^{1}\left[\begin{array}{l}\left\langle D_{\xi}f\left(x,\bar{u}+t\varepsilon\phi,\nabla\bar{u}+t\varepsilon\nabla\phi\right),\nabla\phi\right\rangle \\ \quad+\left\langle D_{u}f\left(x,\bar{u}+t\varepsilon\phi,\nabla\bar{u}+t\varepsilon\nabla\phi\right),\phi\right\rangle\end{array}\right]\mathrm{d}t
$$

Clearly, all we need to prove is that we have

$$
0=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left(I\left[\bar{u}+\varepsilon\phi\right]-I\left[\bar{u}\right]\right)=\lim_{\varepsilon\to0}\int_{\Omega}g\left(x,\varepsilon\right)\mathrm{d}x=\int_{\Omega}\lim_{\varepsilon\to0}g\left(x,\varepsilon\right)\mathrm{d}x.
$$

This will follow from dominated convergence theorem as soon as we can establish a bound of $g\left(x,\varepsilon\right)$ which is independent of ε and is in $L^{1}\left(\Omega\right)$. Using $(G_{p,\mathrm{cont}})$, we have

$$
\begin{aligned}
&\left|\left\langle D_{u}f\left(x,\bar{u}+t\varepsilon\phi,\nabla\bar{u}+t\varepsilon\nabla\phi\right),\phi\right\rangle\right| \\
&\quad\leq\left|\alpha_{1}\right|\left|\phi\right|+\beta\left|\bar{u}+t\varepsilon\phi\right|^{p-1}\left|\phi\right|+\beta\left|\nabla\bar{u}+t\varepsilon\nabla\phi\right|^{p-1}\left|\phi\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\left\langle D_{\xi}f\left(x,\bar{u}+t\varepsilon\phi,\nabla\bar{u}+t\varepsilon\nabla\phi\right),\nabla\phi\right\rangle\right| \\
&\quad\leq\left|\alpha_{2}\right|\left|\nabla\phi\right|+\beta\left|\bar{u}+t\varepsilon\phi\right|^{p-1}\left|\nabla\phi\right|+\beta\left|\nabla\bar{u}+t\varepsilon\nabla\phi\right|^{p-1}\left|\nabla\phi\right|.
\end{aligned}
$$

From this, it is easy to establish the uniform L^{1} bound. We just show how to estimate the term coming from the last summand above. Using Young's inequality and the triangle inequality, we have

$$
\begin{aligned}
\left|\int_{0}^{1}\left|\nabla\bar{u}+t\varepsilon\nabla\phi\right|^{p-1}\left|\nabla\phi\right|\mathrm{d}t\right| &\leq c\int_{0}^{1}\left(\left|\nabla\bar{u}+t\varepsilon\nabla\phi\right|^{p}+\left|\nabla\phi\right|^{p}\right)\mathrm{d}t \\
&\leq c\int_{0}^{1}\left(\left|\nabla\bar{u}\right|^{p}+\left|t\varepsilon\nabla\phi\right|^{p}+\left|\nabla\phi\right|^{p}\right)\mathrm{d}t.
\end{aligned}
$$

Now since we are interested in $\varepsilon\to0$, we can assume $\left|\varepsilon\right|\leq1$. So we deduce from the last inequality,

$$
\begin{aligned}
&\left|\int_{0}^{1}\left|\nabla\bar{u}+t\varepsilon\nabla\phi\right|^{p-1}\;\left|\nabla\phi\right|\mathrm{d}t\right| \\
&\qquad\leq c\int_{0}^{1}\left(\left|\nabla\bar{u}\right|^{p}+\left|t\varepsilon\right|^{p}\left|\nabla\phi\right|^{p}+\left|\nabla\phi\right|^{p}\right)\mathrm{d}t \\
&\qquad\leq c\int_{0}^{1}\left(\left|\nabla\bar{u}\right|^{p}+\left|\nabla\phi\right|^{p}+\left|\nabla\phi\right|^{p}\right)\mathrm{d}t \\
&\qquad\leq c\left(\left|\nabla\bar{u}\right|^{p}+2\left|\nabla\phi\right|^{p}\right).
\end{aligned}
$$

Now the RHS clearly is in $L^1(\Omega)$ since $\nabla\bar{u}, \nabla\phi \in L^p\left(\Omega; \mathbb{R}^{N\times n}\right)$. Other terms can be estimated in a similar manner. This completes the proof. $\square$

1.6 Glimpses of the Vectorial Calculus of Variations

1.6.1 Necessity of convexity and the vectorial calculus of variations

Necessary condition for wlsc

In general, for sequential weak lower semicontinuity theorems, convexity of the map $\xi \mapsto f(x,u,\xi)$ plays a crucial role. We have already seen that this is sufficient for sequential weak lower semicontinuity assuming the usual lower bounds. Is this a necessary condition for wlsc?

If either $n = 1$ or $N = 1$, this is indeed necessary as well. However, this is far from the case when $n, N \geq 2$. This case is usally referred to the vectorial calculus of variations (or the vectorial case in the calculus of variations).

We do not have enough time left in the course to prove this result. So we shall only state the result.

Theorem 4 (Necessary condition for wlsc)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(x,u,\xi)$ be a Carathéodory function satisfying*

$$|f(x,u,\xi)| \leq a(x) + b(u,\xi)$$

for a.e. $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$, where $a, b \geq 0$, $a \in L^1(\mathbb{R}^n)$ and $b \in C\left(\mathbb{R}^N \times \mathbb{R}^{N\times n}\right)$. Let

$$I[u] = I[u,\Omega] := \int_\Omega f(x,u(x),\nabla u(x))\ \mathrm{d}x$$

and suppose there exists $u_0 \in W^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ such that $I[u_0,\Omega] < \infty$. If I is sequentially weakly $*$ lower semicontinuous in $W^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$, then

$$\frac{1}{|D|}\int_D f(x_0,u_0,\xi_0 + \nabla\phi(y))\ \mathrm{d}y \geq f(x_0,u_0,\xi_0)$$

for every bounded open set $D \subset \mathbb{R}^n$, for a.e. $x_0 \in \Omega$, for every $(u_0,\xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and for every $\phi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$.

Quasiconvexity

The necessary condition above was introduced by Morrey. He also showed that under some standard grwoth assumptions, this is also sufficient.

Definition 5 (Quasiconvexity). *Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(x,u,\xi)$ *be a Carathéodory function.* f *is said to be* ***quasiconvex*** *if it satisfies*

$$\frac{1}{|D|}\int_D f\left(x_0, u_0, \xi_0 + \nabla\phi(y)\right)\,\mathrm{d}y \geq f\left(x_0, u_0, \xi_0\right)$$

for every bounded open set $D \subset \mathbb{R}^n$, *for a.e.* $x_0 \in \Omega$, *for every* $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *and for every* $\phi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$.

Let us now show in a simple setting that

$$\text{convexity} \Rightarrow \text{quasiconvexity.}$$

Proposition 6 (convexity implies quasiconvexity). *Let* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(\xi)$ *be continuous. Then we have*

$$f \text{ convex} \qquad \Rightarrow \qquad f \text{ quasiconvex.}$$

Proof. Note that for any bounded open set $D \subset \mathbb{R}^n$ and any $\phi \in W_0^{1,\infty}\left(D;\mathbb{R}^N\right)$, integrating by parts we deduce,

$$\int_D \frac{\partial \phi^i}{\partial x_\alpha}(y)\,\mathrm{d}y = -\int_D \phi^i(y)\frac{\partial}{\partial x_\alpha}(1)\,\mathrm{d}y = 0$$

for every $1 \leq i \leq N$ and every $1 \leq \alpha \leq n$. Thus, we obtain

$$\frac{1}{|D|}\int_D \nabla\phi(y)\,\mathrm{d}y = 0.$$

Since f is convex, by Jensen's inequality, for any $\xi_0 \in \mathbb{R}^{N\times n}$, we deduce

$$\frac{1}{|D|}\int_D f\left(\xi_0 + \nabla\phi(y)\right)\,\mathrm{d}y \geq f\left(\frac{1}{|D|}\int_D \left[\xi_0 + \nabla\phi(y)\right]\,\mathrm{d}y\right) = f\left(\xi_0\right).$$

This proves f is quasiconvex. ☐

Rank one convexity

However, quasiconvexity generally is hard to check. There is a pointwise condition that is implied by quasiconvexity.

Definition 7 (Rank one convexity). *A function* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(\xi)$ *is called* ***rank one convex*** *if for every* $a \in \mathbb{R}^n$, *every* $b \in \mathbb{R}^N$ *and every* $\xi \in \mathbb{R}^{N\times n}$, *the function*

$$g(t) := f(\xi + ta \otimes b)$$

is convex in t.

Note that for an $N \times n$ matrix X,

$$\operatorname{rank}(X) = 1 \text{ if and only if } X = a \otimes b$$

for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}^N$. Thus this is **convexity along rank one matrices**.

It can be proved that

$$f \text{ quasiconvex} \qquad \Rightarrow \qquad f \text{ rank one convex .}$$

However, neither quasiconvexity nor rank one convexity implies convexity and both are significantly weaker than convexity as soon as $n, N \geq 2$. In the next subsection, we discuss a particularly important example which is nonconvex but is rank one convex an quasiconvex.

1.6.2 The determinant

Rank one convexity of the determinant

Now we give an example of a function which is rank one convex but not convex.

Example 8. *Let $n = N = 2$. Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as*

$$f(\xi) = \det \xi.$$

Then f is rank one convex but not convex.

Indeed, we have

$$\det \begin{pmatrix} \xi_{11} + ta_1b_1 & \xi_{12} + ta_1b_2 \\ \xi_{21} + ta_2b_1 & \xi_{22} + ta_2b_2 \end{pmatrix}$$
$$= (\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) + t\left(a_2b_2\xi_{11} + a_1b_1\xi_{22} - a_2b_1\xi_{12} - a_1b_2\xi_{21}\right),$$

since the terms that are quadratic in t are the same, namely $t^2a_1a_2b_1b_2$, but appear with opposite signs and hence cancel in each other. But this is clearly affine in t. Consequently, $f(\xi) = \det \xi$ is rank one affine (both f and $-f$ are rank one convex). But clearly, for any $\lambda \in (0, 1)$,

$$\lambda(1-\lambda) = \det \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} > \lambda \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (1-\lambda)\det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

So $\det \xi$ can not be convex.

Weak continuity of the determinants

Now we shall show that the determinant is weakly continuous.

Proposition 9. *Let $\Omega \subset \mathbb{R}^2$. Let $\{u_s\}_{s\geq 1} \subset W^{1,p}\left(\Omega, \mathbb{R}^2\right)$ such that*

$$u_s \rightharpoonup u \qquad \text{in } W^{1,p}\left(\Omega, \mathbb{R}^2\right)$$

for some $2 < p < \infty$. Then up to the extraction of a subsequence,

$$\det \nabla u_s \rightharpoonup \det \nabla u \qquad \text{in } L^{\frac{p}{2}}(\Omega).$$

Proof. By Hölder inequality, it is easy to show that $\det \nabla u_s$ is uniformly bounded in $L^{\frac{p}{2}}(\Omega)$ and thus up to the extraction of a subsequence, this converges weakly in $L^{\frac{p}{2}}$ to a weak limit. So we just have to identify the weak limit.

So it is enough to show that for every $\psi \in C_c^{\infty}(\Omega)$, we have,

$$\int_{\Omega} \det \nabla u_s(x)\, \psi(x)\ \mathrm{d}x \to \int_{\Omega} \det \nabla u(x)\, \psi(x)\ \mathrm{d}x.$$

Now if u_s is C^2, we have

$$\begin{aligned}
\det \nabla u_s &= \frac{\partial u_s^1}{\partial x_1}\frac{\partial u_s^2}{\partial x_2} - \frac{\partial u_s^1}{\partial x_2}\frac{\partial u_s^2}{\partial x_1} \\
&= \frac{\partial}{\partial x_1}\left(u_s^1 \frac{\partial u_s^2}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-u_s^1 \frac{\partial u_s^2}{\partial x_1}\right) \\
&= \operatorname{div}\left(u_s^1 \frac{\partial u_s^2}{\partial x_2}, -u_s^1 \frac{\partial u_s^2}{\partial x_1}\right).
\end{aligned}$$

So integrating by parts, we obtain

$$\begin{aligned}
\int_{\Omega} \det \nabla u_s(x)\, \psi(x)\ \mathrm{d}x &= \int_{\Omega} \operatorname{div}\left(u_s^1 \frac{\partial u_s^2}{\partial x_2}, -u_s^1 \frac{\partial u_s^2}{\partial x_1}\right)(x)\, \psi(x)\ \mathrm{d}x \\
&= -\int_{\Omega} \left\langle \left(u_s^1 \frac{\partial u_s^2}{\partial x_2}, -u_s^1 \frac{\partial u_s^2}{\partial x_1}\right)(x), \nabla \psi(x) \right\rangle \mathrm{d}x
\end{aligned}$$

The last identity is truw for u_s in $W^{1,p}$ as well, by density. Now we claim

$$\int_{\Omega} \left\langle \left(u_s^1 \frac{\partial u_s^2}{\partial x_2}, -u_s^1 \frac{\partial u_s^2}{\partial x_1}\right), \nabla \psi \right\rangle \to \int_{\Omega} \left\langle \left(u^1 \frac{\partial u^2}{\partial x_2}, -u^1 \frac{\partial u^2}{\partial x_1}\right), \nabla \psi \right\rangle.$$

This is enough to prove the result by another integration by parts. Now we show

$$\int_{\Omega} u_s^1(x) \frac{\partial u_s^2}{\partial x_2}(x) \frac{\partial \psi}{\partial x_1}(x)\ \mathrm{d}x \to \int_{\Omega} u^1(x) \frac{\partial u^2}{\partial x_2}(x) \frac{\partial \psi}{\partial x_1}(x)\ \mathrm{d}x.$$

By Rellich-Kondrachov, $u_s \to u$ strongly in L^p. Thus, we have,

$$\begin{aligned}
&\int_{\Omega} \left(u_s^1 \frac{\partial u_s^2}{\partial x_2}\frac{\partial \psi}{\partial x_1} - u^1 \frac{\partial u^2}{\partial x_2}\frac{\partial \psi}{\partial x_1}\right) \mathrm{d}x \\
&\quad = \int_{\Omega} \left(u_s^1 - u^1\right) \frac{\partial u_s^2}{\partial x_2}\frac{\partial \psi}{\partial x_1}\ \mathrm{d}x + \int_{\Omega} u^1 \left(\frac{\partial u_s^2}{\partial x_2} - \frac{\partial u^2}{\partial x_2}\right)\frac{\partial \psi}{\partial x_1}\ \mathrm{d}x
\end{aligned}$$

The second term converges to zero by definition of weak convergence in L^p and the fact that

$$\nabla u_s \rightharpoonup \nabla u \qquad \text{in } L^p.$$

Now we can estimate

$$\left| \int_{\Omega} \left(u_s^1 - u^1\right) \frac{\partial u_s^2}{\partial x_2}\frac{\partial \psi}{\partial x_1}\ \mathrm{d}x \right| \leq \left\| u_s^1 - u^1 \right\|_{L^p} \left\| \frac{\partial u_s^2}{\partial x_2} \right\|_{L^p} \left\| \nabla \psi \right\|_{L^{\infty}}.$$

The RHS clearly goes to zero as ∇u_s is uniformly bounded in L^p and the strong convergence $u_s \to u$ in L^p. This completes the proof. $\square$

Note that exactly the same proof establishes the following.

Proposition 10. *Let* $\Omega \subset \mathbb{R}^2$. *Let* $\{u_s\}_{s\geq 1} \subset W^{1,2}\left(\Omega, \mathbb{R}^2\right)$ *such that*

$$u_s \rightharpoonup u \qquad \text{in } W^{1,2}\left(\Omega, \mathbb{R}^2\right).$$

Then up to the extraction of a subsequence, $\{\det \nabla u_s\}_{s\geq 1}$ *converges in the sense of distributions, written as*

$$\det \nabla u_s \rightharpoonup \det \nabla u \qquad \text{in } \mathcal{D}^{'}(\Omega),$$

which explicitly means that for every $\psi \in C_c^{\infty}(\Omega)$, *we have,*

$$\int_{\Omega} \det \nabla u_s(x)\, \psi(x)\ \mathrm{d}x \to \int_{\Omega} \det \nabla u(x)\, \psi(x)\ \mathrm{d}x.$$

Remark 11. *In this case, i.e. when* $p = 2$, *we* ***can not conclude*** *that*

$$\det \nabla u_s \rightharpoonup \det \nabla u \qquad \text{in } L^1(\Omega).$$

It is false in general. Note that since L^1 *non-reflexive, although* $\{\det \nabla u_s\}_{s\geq 1}$ *is uniformly bounded in* L^1, *we can ascertain the existence of a subsequence that converges weakly in* L^1.