Introduction to the Calculus of Variations
Lecture Notes
Lecture 18

Swarnendu Sil

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Chapter 1

Direct methods

1.1 Dirichlet Integral

1.2 Integrands depending only on the gradient

1.3 Integrands with x dependence

1.4 Integrands with x and u dependence

Unfortunately, our hypotheses still leave out important problems. For example, the PDE

$$\Delta u = f \qquad \text{in } \Omega$$

where $f \not\equiv 0$. Indeed, the energy functional is

$$I[u] = \int_\Omega \left[\frac{1}{2} |\nabla u(x)|^2 + \langle f(x), u(x) \rangle \right] \, dx.$$

This depends not only on x, but also explicitly on u. However, here at least the dependence on u is linear. The functional

$$I[u] = \int_\Omega \left[\frac{1}{2} |\nabla u(x)|^2 + \frac{\lambda}{2} |u(x)|^2 \right] \, dx,$$

which corresponds to the eigenvalue problem

$$\Delta u = \lambda u \qquad \text{in } \Omega$$

is a more general and an important example.

1.4.1 Weak lower semicontinuity

Scorza-Dragoni theorem

Proving a weak lower semicontinuity result in the general case is quite delicate and we need some preparations. First, we need a generalization of the classical Lusin's theorem for Carathéodory functions. Measurable dependence on x creates difficulties in handling, so we improve measurability to continuity at the cost of leaving out a set of controlled small measure.

Theorem 1 (Scorza-Dragoni)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable and let $S \subset \mathbb{R}^M$ be **compact**. Let $f : \Omega \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function. Then for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that*

$$|\Omega \setminus K_\varepsilon| < \varepsilon \qquad \text{and} \qquad f\big|_{K_\varepsilon \times S} \text{ is continuous }.$$

Proof. For any natural number $i \in \mathbb{N}$, set

$$\omega_i(x) := \sup\left\{ |f(x,u) - f(x,v)| : u, v \in S, |u - v| \leq \frac{1}{i} \right\}.$$

Since f is Carathéodory, we have $\omega_i(x) \to 0$ for a.e. $x \in \Omega$. Thus by Egoroff (or Egorov) theorem, for any $\varepsilon > 0$, there exists a compact set $K^1_\varepsilon \subset \Omega$ such that

$$\omega_i(x) \to 0 \quad \text{uniformly on } K^1_\varepsilon \qquad \text{and} \qquad \left|\Omega \setminus K^1_\varepsilon\right| < \frac{\varepsilon}{2}.$$

This implies that for any $\eta > 0$ and any $u \in S$, there exists a $\delta_1 = \delta_1(\eta, u) > 0$ such that for every $x \in K^1_\varepsilon$ and $v \in S$,

$$|u - v| < \delta_1 \quad \Rightarrow \quad |f(x,u) - f(x,v)| < \frac{\eta}{4}. \tag{1.1}$$

Now we choose a sequence $\{u_i\}_{i \geq 1}$ which is dense in S. Now, applying Lusin (or Luzin) theorem, for each $i \in \mathbb{N}$, we can find a compact set $K_i \subset \Omega$ so that

$$x \mapsto f(x, u_i) \text{ is continuous in } K_i \qquad \text{and} \qquad |\Omega \setminus K_i| < \frac{\varepsilon}{2^{i+1}}.$$

We set

$$K^2_\varepsilon := \bigcap_{i=1}^{\infty} K_i.$$

Then we have

$$\left|\Omega \setminus K^2_\varepsilon\right| \leq \sum_{i=1}^{\infty} |\Omega \setminus K_i| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2}$$

and we have

$$x \mapsto f(x, u_i) \text{ is continuous in } K^2_\varepsilon \text{ for all } i \in \mathbb{N}.$$

Thus, for any $\eta > 0$, any $x \in K_\varepsilon^2$ and any u_i, there exists a $\delta_2 = \delta_2(x, \eta, u_i) > 0$ such that for every $y \in K_\varepsilon^2$,

$$|x - y| < \delta_2 \quad \Rightarrow \quad |f(x, u_i) - f(y, u_i)| < \frac{\eta}{4}. \tag{1.2}$$

Now we set

$$K_\varepsilon = K_\varepsilon^1 \cap K_\varepsilon^2.$$

Clearly, this implies

$$|\Omega \setminus K_\varepsilon| \leq \left|\Omega \setminus K_\varepsilon^1\right| + \left|\Omega \setminus K_\varepsilon^2\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now we show that f is continuous on $K_\varepsilon \times S$. Let $x \in K_\varepsilon$ and $u \in S$. We want to show that given any $\eta > 0$, there exists a $\delta = \delta(\eta, x, u) > 0$ such that for any $y \in K_\varepsilon$ and $v \in S$,

$$|x - y| + |u - v| < \delta \quad \Rightarrow \quad |f(x, u) - f(y, v)| < \eta.$$

Now we first choose $\delta_1 = \delta_1(\eta, u)$ as in (1.1) and then by density, pick u_i such that

$$|u - u_i| < \delta_1.$$

Now once we have picked u_i in this way, we can choose $\delta_2 = \delta_2(x, \eta, u_i)$ as in (1.2) and set

$$\delta = \delta(x, u, \eta) := \min\left\{\delta_1(\eta, u), \delta_2(x, \eta, u_i)\right\}.$$

Now if

$$|x - y| + |u - v| < \delta$$

we have

$$\begin{aligned}
&|f(x, u) - f(y, v)| \\
&\quad \leq |f(x, u) - f(x, u_i)| + |f(x, u_i) - f(y, u_i)| + |f(y, u_i) - f(y, u)| \\
&\qquad\qquad + |f(y, u) - f(y, v)| \\
&\quad < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} = \eta,
\end{aligned}$$

where we have used that the first and third summand is less than $\eta/4$ since $|u - u_i| < \delta_1$, the fourth summand is less than $\eta/4$ since $|u - v| < \delta_1$ and the second summand is less than $\eta/4$ since $|x - y| < \delta_2$. This completes the proof of the theorem. □

Weak lower semicontinuity: general case

Theorem 2 (Weak lower semicontinuity: the general case)**.** *Let $n \geq 2, N \geq 1$ be integers and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$, $f = f(x, u, \xi)$ be a Carathéodory function satisfying*

$$f(x, u, \xi) \geq \langle a(x), \xi \rangle + b(x) + c\,|u|^r$$

for a.e. $x \in \Omega$, *for every* $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *for some* $a \in L^{p'}\left(\Omega; \mathbb{R}^{N\times n}\right)$, $b \in L^1(\Omega)$, $c \in \mathbb{R}$, $1 \leq r < \frac{np}{n-p}$ *if* $1 \leq p < n$ *and* $1 \leq r < \infty$ *if* $n \leq p < \infty$. *Let*

$$I[u] := \int_\Omega f(x, u(x), \nabla u(x))\ \mathrm{d}x.$$

Let $\xi \mapsto f(x, u, \xi)$ *be convex for a.e.* $x \in \Omega$ *and for every* $u \in \mathbb{R}^N$. *Let* $u_s \rightharpoonup u$ *weakly in* $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$. *Then we have*

$$\liminf_{s\to\infty} I[u_s] \geq I[u].$$

Proof. We begin by noting that we can assume $f \geq 0$. Indeed, we can replace f by

$$g(x, u, \xi) := f(x, u, \xi) - \langle a(x), \xi\rangle + b(x) + c|u|^r.$$

By our assumption on the exponent r and Rellich-Kondrachov theorem, we know

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}\left(\Omega; \mathbb{R}^N\right) \qquad \Rightarrow \qquad u_s \to u \quad \text{in } L^r\left(\Omega; \mathbb{R}^N\right).$$

This last convergence implies

$$\|u_s\|_{L^r(\Omega;\mathbb{R}^N)} \to \|u\|_{L^r(\Omega;\mathbb{R}^N)}.$$

Thus, we easily deduce

$$\begin{aligned}\liminf_{s\to\infty} \int_\Omega g(x, u_s(x), \nabla u_s(x))\ \mathrm{d}x - \int_\Omega g(x, u(x), \nabla u(x))\ \mathrm{d}x \\ = \liminf_{s\to\infty} \int_\Omega f(x, u_s(x), \nabla u_s(x))\ \mathrm{d}x - \int_\Omega f(x, u(x), \nabla u(x))\ \mathrm{d}x.\end{aligned}$$

Thus, it is enough to prove the theorem with the additional assumption that $f \geq 0$.

Now our task is to reduce the proof to the previous case, i.e. integrands depending only on x and ξ, by 'freezing' u. As before, let

$$L := \liminf_{s\to\infty} \int_\Omega f(x, u_s(x), \nabla u_s(x))\ \mathrm{d}x$$

and passing to a subsequence if necessary, we can assume

$$L := \lim_{s\to\infty} \int_\Omega f(x, u_s(x), \nabla u_s(x))\ \mathrm{d}x.$$

Fix $\varepsilon > 0$. We want to show

Claim 3. *There exists a measurable set* $\Omega_\varepsilon \subset \Omega$ *and a subsequence* $\{s_j\}_{j\geq 1}$ *with* $s_j \to +\infty$ *such that*

$$|\Omega \setminus \Omega_\varepsilon| < \varepsilon,$$

$$\int_{\Omega_\varepsilon} \left| f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) - f\left(x, u(x), \nabla u_{s_j}(x)\right)\right|\ \mathrm{d}x < \varepsilon|\Omega|$$

for every $j \geq 1$.

Let us first complete the proof assuming the claim. Set

$$g(x,\xi) := \mathbb{1}_{\Omega_\varepsilon}(x) f(x, u(x), \xi).$$

By the wlsc theorem for integrands with x dependence, we get

$$\liminf_{j\to\infty} \int_{\Omega_\varepsilon} f\left(x, u(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \geq \int_{\Omega_\varepsilon} f(x, u(x), \nabla u(x)) \,\mathrm{d}x.$$

But since $f \geq 0$, we have

$$\int_\Omega f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \geq \int_{\Omega_\varepsilon} f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x$$

By the claim, we deduce

$$\begin{aligned}
&\int_{\Omega_\varepsilon} f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \\
&\quad\geq \int_{\Omega_\varepsilon} f\left(x, u(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \\
&\qquad - \int_{\Omega_\varepsilon} \left|f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) - f\left(x, u(x), \nabla u_{s_j}(x)\right)\right| \,\mathrm{d}x \\
&\quad\geq \int_{\Omega_\varepsilon} f\left(x, u(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x - \varepsilon |\Omega|.
\end{aligned}$$

Combining the last three inequalities, we have

$$\begin{aligned}
L &= \liminf_{j\to\infty} \int_\Omega f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \\
&\geq \liminf_{j\to\infty} \int_{\Omega_\varepsilon} f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x \\
&\geq \liminf_{j\to\infty} \int_{\Omega_\varepsilon} f\left(x, u(x), \nabla u_{s_j}(x)\right) \,\mathrm{d}x - \varepsilon |\Omega| \\
&\geq \int_{\Omega_\varepsilon} f(x, u(x), \nabla u(x)) \,\mathrm{d}x - \varepsilon |\Omega| \\
&= \int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) f(x, u(x), \nabla u(x)) \,\mathrm{d}x - \varepsilon |\Omega|.
\end{aligned}$$

Note that by monotone convergence

$$\int_\Omega \mathbb{1}_{\Omega_\varepsilon}(x) f(x, u(x), \nabla u(x)) \,\mathrm{d}x \to \int_\Omega f(x, u(x), \nabla u(x)) \,\mathrm{d}x$$

as $\varepsilon \to 0$. So letting $\varepsilon \to 0$, we prove the conclusion. Now it remains to prove the Claim 3.

Proof of Claim 3

Fix $\varepsilon_j > 0$ for now. For any $h \in L^q\left(\Omega;\mathbb{R}^N\right)$ for some $1 \leq q < \infty$, from the Chebyshev's inequality we deduce the following estimate

$$|\{x \in \Omega : |h(x)| \geq t\}| \leq \frac{1}{t^q}\int_{|h|\geq t} |h(x)|^q \, \mathrm{d}x \leq \frac{1}{t^q}\|h\|^q_{L^q(\Omega;\mathbb{R}^N)}.$$

Thus, we can choose a number $M_{\varepsilon_j} > 0$ large enough and independent of s such that

$$\left|\Omega \setminus \Omega^1_{\varepsilon_j,s}\right| < \frac{\varepsilon_j}{3},$$

where

$$\Omega^1_{\varepsilon_j,s} := \left\{x \in \Omega : |u(x)|, |u_s(x)|, |\nabla u_s(x)| < M_{\varepsilon_j} \text{ for every } s \geq 1\right\}.$$

Now since f is Carathéodory, applying the Scorza-Dragoni theorem, we find a compact set $\Omega^2_{\varepsilon_j,s} \subset \Omega^1_{\varepsilon_j,s}$ such that

$$\left|\Omega^1_{\varepsilon_j,s} \setminus \Omega^2_{\varepsilon_j,s}\right| < \frac{\varepsilon_j}{3} \quad \text{and } f\big|_{\Omega^2_{\varepsilon_j,s}\times S_{\varepsilon_j}} \text{ is continuous,}$$

where

$$S_\varepsilon := \left\{(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n} : |u|, |\xi| < M_{\varepsilon_j}\right\}.$$

Hence, by continuity, there exists $\delta(\varepsilon_j) > 0$ such that

$$|u - v| < \delta(\varepsilon_j) \qquad \Rightarrow \qquad |f(x,u,\xi) - f(x,v,\xi)| < \varepsilon_j$$

for all $x \in \Omega^2_{\varepsilon_j,s}$, for all $|u|, |v|, |\xi| < M_{\varepsilon_j}$. But by the convergence

$$u_s \to u \text{ strongly in } L^r\left(\Omega;\mathbb{R}^N\right)$$

and the Chebyshev's inequality, we can find $s_{\varepsilon_j} \in \mathbb{N}$ such that if

$$\Omega^3_{\varepsilon_j,s} := \{x \in \Omega : |u_s(x) - u(x)| < \delta(\varepsilon_j)\},$$

then

$$\left|\Omega \setminus \Omega^3_{\varepsilon_j,s}\right| < \frac{\varepsilon_j}{3} \qquad \text{for all } s \geq s_{\varepsilon_j}.$$

Now we set

$$\Omega_{\varepsilon_j,s_{\varepsilon_j}} := \Omega^2_{\varepsilon_j,s} \cap \Omega^3_{\varepsilon_j,s}.$$

Clearly, we have

$$\left|\Omega \setminus \Omega_{\varepsilon_j,s_{\varepsilon_j}}\right| \leq \left|\Omega \setminus \Omega^2_{\varepsilon_j,s}\right| + \left|\Omega \setminus \Omega^3_{\varepsilon_j,s}\right| < \frac{2\varepsilon_j}{3} + \frac{\varepsilon_j}{3} = \varepsilon_j.$$

Also, we have,

$$\int_{\Omega_{\varepsilon_j,s_{\varepsilon_j}}} |f(x,u_s(x),\nabla u_s(x)) - f(x,u(x),\nabla u_s(x))| \, \mathrm{d}x$$
$$< \varepsilon_j \left|\Omega_{\varepsilon_j,s_{\varepsilon_j}}\right| \leq \varepsilon_j |\Omega|$$

for every $s \geq s_{\varepsilon_j}$. Now we choose $\varepsilon_j := 2^{-j}\varepsilon$ for $j \geq 1$. For every $j \geq 1$, we pick an natural number $s_j \geq s_{\varepsilon_j}$ such that $s_j \to \infty$ as $j \to \infty$. Finally, we set

$$\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_{\varepsilon_j, s_{\varepsilon_j}}.$$

Thus, we have

$$|\Omega \setminus \Omega_\varepsilon| \leq \sum_{j=1}^{\infty} \left| \Omega \setminus \Omega_{\varepsilon_j, s_{\varepsilon_j}} \right| < \sum_{j=1}^{\infty} \varepsilon_j = \varepsilon \left(\sum_{j=1}^{\infty} \frac{1}{2^j} \right) = \varepsilon.$$

Also, for every $j \geq 1$, we have

$$\int_{\Omega_\varepsilon} \left| f\left(x, u_{s_j}(x), \nabla u_{s_j}(x)\right) - f\left(x, u(x), \nabla u_{s_j}(x)\right) \right| \, \mathrm{d}x$$
$$< \varepsilon_j |\Omega| < \varepsilon |\Omega|.$$

This proves the claim and finishes the proof of the theorem. □

1.4.2 Existence of minimizer: the general case

Theorem 4. *Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ be given. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function satisfying*

$$f(x, u, \xi) \geq c_1 |\xi|^p + c_2 |u|^q + b(x)$$

for a.e. $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$, $b \in L^1(\Omega)$ and $1 \leq q < p$. Assume $\xi \mapsto f(x, u, \xi)$ be convex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^N$. Let

$$I[u] := \int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}x.$$

If $I[u_0] < \infty$, then the following problem

$$\inf \left\{ I[u] : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\} = m$$

admits a minimizer. If $(u, \xi) \mapsto f(x, u, \xi)$ is strictly convex for a.e. $x \in \Omega$, then the minimizer is unique.

Proof. Let $\{u_s\}_{s \geq 1}$ be a minimizing sequence. Then we have,

$$\begin{aligned} m + 1 &\geq I[u_s] \\ &\geq c_1 \int_\Omega |\nabla u_s(x)|^p \, \mathrm{d}x - |c_2| \int_\Omega |u_s(x)|^q \, \mathrm{d}x - \int_\Omega |b(x)| \, \mathrm{d}x \\ &= c_1 \|\nabla u_s\|^p_{L^p(\Omega; \mathbb{R}^{N \times n})} - |c_2| \|u_s\|^q_{L^q(\Omega; \mathbb{R}^N)} - \|b\|_{L^1(\Omega)}. \end{aligned}$$

By Hölder inequality, we have,

$$\|u_s\|_{L^q(\Omega;\mathbb{R}^N)}^q \leq |\Omega|^{\frac{p-q}{p}} \|u_s\|_{L^p(\Omega;\mathbb{R}^N)}^q .$$

Thus, there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$\begin{aligned} m+1 &\geq c_1 \|\nabla u_s\|_{L^p(\Omega;\mathbb{R}^{N\times n})}^p - \gamma_1 \|u_s\|_{L^p(\Omega;\mathbb{R}^N)}^q - \gamma_2 \\ &\geq c_1 \|\nabla u_s\|_{L^p(\Omega;\mathbb{R}^{N\times n})}^p - \gamma_1 \|u_s\|_{W^{1,p}(\Omega;\mathbb{R}^N)}^q - \gamma_2. \end{aligned}$$

By Poincaré inequality, we can find constants $\gamma_3, \gamma_4, \gamma_5 > 0$ such that

$$m+1 \geq \gamma_3 \|u_s\|_{W^{1,p}(\Omega;\mathbb{R}^N)}^p - \gamma_4 \|u_0\|_{W^{1,p}(\Omega;\mathbb{R}^N)}^p - \gamma_1 \|u_s\|_{W^{1,p}(\Omega;\mathbb{R}^N)}^q - \gamma_5.$$

Since $1 \leq q < p$, we can find constants $\gamma_7, \gamma_8 > 0$ such that

$$m+1 \geq \gamma_7 \|u_s\|_{W^{1,p}(\Omega;\mathbb{R}^N)}^p - \gamma_8.$$

This implies $\{u_s\}_{s\geq 1}$ is uniformly bounded in $W^{1,p}$. The rest follows the same way as before using the weak lower semicontinuity theorem. The inequality in the hypothesis can be easily verified from the coercivity inequality by taking $a \equiv 0$, $r = q$ and the same b. This completes the proof. □