Introduction to the Calculus of Variations
Lecture Notes
Lecture 17

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Chapter 1

Direct methods

1.1 Dirichlet Integral

1.2 Integrands depending only on the gradient

Armed with our experience with the Dirichlet integral, we now move on to more general integrals. We begin with the case where the integrand is a fairly general convex function of the gradient. As we have already seen, perhaps the most important property for applying the direct methods is the sequential weak lower semicontinuity.

1.2.1 Weak lower semicontinuity with no lower order terms

Mazur lemma

We begin by recalling a standard result in functional analysis whose proof can be found in most textbooks, see for example, Corollary 3.8 in [1].

Lemma 1 (Mazur lemma)**.** *Let $(X, \|\cdot\|)$ be a normed space and let*

$$x_s \rightharpoonup x \qquad \text{in } X.$$

Then there exists a sequence $\{y_\mu\}_{\mu \geq 1} \subset \operatorname{co}\{x_s\}_{s\geq 1}$ such that

$$y_\mu \to x \qquad \text{in } X.$$

More precisely, for every μ, there exist an integer m_μ and

$$\alpha_\mu^i > 0 \quad \text{with} \quad \sum_{i=1}^{m_\mu} \alpha_\mu^i = 1$$

such that

$$y_\mu = \sum_{i=1}^{m_\mu} \alpha_\mu^i x_i \qquad \text{and} \qquad \|y_\mu - x\|_X \to 0 \quad \text{as} \quad \mu \to \infty.$$

Theorem 2 (sequential weak lower semicontinuity with no lower order terms). *Let $n \geq 2, N \geq 1$ be integers and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ be continuous. Let*

$$I[u] := \int_\Omega f(\nabla u(x))\ \mathrm{d}x.$$

Let $\xi \mapsto f(\xi)$ be convex and

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}\left(\Omega; \mathbb{R}^N\right).$$

Then we have

$$\liminf_{s\to\infty} I[u_s] \geq I[u].$$

Proof. We divide the proof into three steps.

Reduction to positive integrands We first show we can assume $f \geq 0$. Since f is convex, there exists a vector $\theta \in \mathbb{R}^{N\times n}$ such that

$$f(\xi) \geq f(0) + \langle \theta, \xi \rangle \qquad \text{for all } \xi \in \mathbb{R}^{N\times n}.$$

In Problem sheet 3, we talked about *subgradients* of a convex function. Here θ is nothing but a subgradient of f at 0, i.e. $\theta \in \partial f(0)$. Now we set

$$g(\xi) := f(\xi) - f(0) - \langle \theta, \xi \rangle.$$

Clearly, $g \geq 0$. Note that since the vector θ, considered as the constant function is in $L^{p'}(\Omega)$, for every sequence

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}\left(\Omega; \mathbb{R}^N\right),$$

we have

$$\int_\Omega \langle \theta, \nabla u_s(x) \rangle\ \mathrm{d}x \to \int_\Omega \langle \theta, \nabla u(x) \rangle\ \mathrm{d}x.$$

This implies that

$$\liminf_{s\to\infty} \int_\Omega g(\nabla u_s(x))\ \mathrm{d}x - \int_\Omega g(\nabla u(x))\ \mathrm{d}x = \liminf_{s\to\infty} \int_\Omega f(\nabla u_s(x))\ \mathrm{d}x - \int_\Omega f(\nabla u(x))\ \mathrm{d}x.$$

Thus, it is enough to prove the theorem with the additional assumption that $f \geq 0$. From now onwards, we keep this assumption in force.

Strong lower semicontinuity Now we prove that we have strong lower semicontinuity, i.e. if we have a sequence

$$v_\mu \to u \quad \text{in } W^{1,p}\left(\Omega; \mathbb{R}^N\right),$$

then up to the extraction of a subsequence, we have

$$\liminf_{\mu\to\infty}\int_\Omega f\left(\nabla v_\mu(x)\right)\ \mathrm{d}x \geq \int_\Omega f\left(\nabla u(x)\right)\ \mathrm{d}x.$$

Indeed the strong convergence in $W^{1,p}$ implies

$$\nabla v_\mu \to \nabla u \quad \text{in } L^p\left(\Omega;\mathbb{R}^{N\times n}\right).$$

But this implies that up to the extraction of a subsequence,

$$\nabla v_\mu \to \nabla u \quad \text{a.e. in } \Omega.$$

Since f is nonnegative, by Fatou's lemma and continuity of f, this gives us

$$\liminf_{\mu\to\infty}\int_\Omega f\left(\nabla v_\mu(x)\right)\ \mathrm{d}x \geq \int_\Omega \liminf_{\mu\to\infty} f\left(\nabla v_\mu(x)\right)\ \mathrm{d}x = \int_\Omega f\left(\nabla u(x)\right)\ \mathrm{d}x.$$

Weak lower semicontinuity Now we finish the proof using the strong lower semicontinuity. Let

$$L := \liminf_{s\to\infty}\int_\Omega f\left(\nabla u_s(x)\right)\ \mathrm{d}x$$

and passing to a subsequence if necessary, we can assume

$$L := \lim_{s\to\infty}\int_\Omega f\left(\nabla u_s(x)\right)\ \mathrm{d}x.$$

Fix $\varepsilon > 0$. Thus, there exists $s_0 = s_0(\varepsilon) \in \mathbb{N}$ such that for every $s \geq s_0$, we have

$$\int_\Omega f\left(\nabla u_s(x)\right)\ \mathrm{d}x \leq L + \varepsilon.$$

Now, for every sequence

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}\left(\Omega;\mathbb{R}^N\right),$$

applying Mazur's lemma, we know there exists a sequence $\{v_\mu\}_{\mu\geq 1}$ such that for every μ, there exist an integer $m_\mu \geq s_0$ and

$$\alpha_\mu^i > 0 \quad \text{with} \quad \sum_{i=s_0}^{m_\mu}\alpha_\mu^i = 1$$

such that

$$v_\mu = \sum_{i=s_0}^{m_\mu}\alpha_\mu^s u_i \qquad \text{and} \qquad v_\mu \to u \quad \text{in } W^{1,p}\left(\Omega;\mathbb{R}^N\right).$$

Now, since f is convex, for every $\mu \geq 1$, we have

$$f\left(\nabla v_\mu\right) = f\left(\sum_{i=s_0}^{m_\mu} \alpha_\mu^i \nabla u_i\right) \leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i f\left(\nabla u_i\right).$$

Thus, for every $\mu \geq 1$, we get

$$\begin{aligned}\int_\Omega f\left(\nabla v_\mu(x)\right)\,\mathrm{d}x &\leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i \int_\Omega f\left(\nabla u_i(x)\right)\,\mathrm{d}x \\ &\leq \sum_{i=s_0}^{m_\mu} \alpha_\mu^i \left(L+\varepsilon\right) = L+\varepsilon.\end{aligned}$$

So finally, we deduce

$$\int_\Omega f\left(\nabla u(x)\right)\,\mathrm{d}x \leq \liminf_{\mu\to\infty} \int_\Omega f\left(\nabla v_\mu(x)\right)\,\mathrm{d}x \leq L+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\int_\Omega f\left(\nabla u(x)\right)\,\mathrm{d}x \leq \liminf_{s\to\infty} \int_\Omega f\left(\nabla u_s(x)\right)\,\mathrm{d}x.$$

This completes the proof. □

Existence of minimizer with no lower order terms

Once the sequential w.l.s.c is established, it is now a routine exercise to prove the following.

Theorem 3. *Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ be given. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$ be continuous, convex and satisfies*

$$f(\xi) \geq c\,|\xi|^p \qquad \text{for all } \xi \in \mathbb{R}^{N\times n} \tag{1.1}$$

for some $c > 0$. Let

$$I[u] := \int_\Omega f\left(\nabla u(x)\right)\,\mathrm{d}x.$$

If $I[u_0] < \infty$, then the following problem

$$\inf\left\{I[u] : u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\} = m$$

admits a minimizer. If f is strictly convex, then the minimizer is unique.

Proof. We first prove the existence part.

(Existence) Let $\{u_s\}_{s\geq 1} \subset u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ be a minimizing sequence, i.e

$$I[u_s] \to m \qquad \text{as } s \to \infty.$$

By virtue of the coercivity assumption (1.1), we deduce

$$\|\nabla u_s\|^p_{L^p(\Omega;\mathbb{R}^{N\times n})} \leq \frac{1}{c} I\left[u_s\right] \qquad \text{for all } s \geq 1. \tag{1.2}$$

Since $u_s - u_0 \in W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ for every $s \geq 1$, by Poincaré inequality, we have

$$\begin{aligned}
\|u_s - u_0\|_{W^{1,p}(\Omega;\mathbb{R}^N)} &\leq c \, \|\nabla u_s - \nabla u_0\|_{L^p(\Omega;\mathbb{R}^{N\times n})} \\
&\leq c \left(\|\nabla u_s\|_{L^p(\Omega;\mathbb{R}^{N\times n})} + \|\nabla u_0\|_{L^p(\Omega;\mathbb{R}^{N\times n})} \right) \\
&\leq c \left(\left(I\left[u_s\right]\right)^{\frac{1}{p}} + \|\nabla u_0\|_{L^p(\Omega;\mathbb{R}^{N\times n})} \right) \\
&\leq c \left(m+1\right)^{\frac{1}{p}} + c \, \|\nabla u_0\|_{L^p} .
\end{aligned}$$

Thus, we have

$$\|u_s\|_{W^{1,p}} \leq \|u_s - u_0\|_{W^{1,p}} + \|u_0\|_{W^{1,p}} \leq c \left(m+1\right)^{\frac{1}{p}} + c \, \|u_0\|_{W^{1,p}} .$$

This proves that $\{u_s\}_{s\geq 1}$ is uniformly bounded in $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$.

Since $1 < p < \infty$, $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ is reflexive and thus the uniform bound implies that there exists $u \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ such that up to the extraction of a subsequence, which we do not relabel, we have

$$u_s \rightharpoonup u \qquad \text{weakly in } W^{1,p}\left(\Omega; \mathbb{R}^N\right).$$

This implies that

$$u_s - u_0 \rightharpoonup u - u_0 \qquad \text{weakly in } W^{1,p}\left(\Omega; \mathbb{R}^N\right).$$

But $u_s - u_0 \in W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ for every $s \geq 1$. Since $W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ is a norm-closed convex subset of $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$, it is also weakly closed and thus the above convergence implies that $u - u_0 \in W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$. Thus, we have established the existence of $u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ such that

$$u_s \rightharpoonup u \qquad \text{weakly in } W^{1,p}\left(\Omega; \mathbb{R}^N\right).$$

But since f is convex, applying Theorem 2, we deduce

$$\liminf_{s\to\infty} I\left[u_s\right] \geq I\left[u\right].$$

But since $u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$, we have

$$m \leq I\left[u\right] \leq \liminf_{s\to\infty} I\left[u_s\right] = m,$$

where the last equality is true by virtue of the fact that u_s is a minimizing sequence. Thus, all the inequalities above must be equalities, showing $I[u] = m$, which means u is a minimizer.

Now we show the uniqueness part.

(Uniqueness) Suppose $\bar{u}$ and $\bar{v}$ are both minimizers. Then let $\bar{w} := \frac{1}{2}(\bar{u} + \bar{v})$. Then we can see, by convexity of f,

$$m \leq I[\bar{w}] \leq \frac{1}{2} I[\bar{u}] + \frac{1}{2} I[\bar{v}] \leq m.$$

So $\bar{w}$ is also a minimizer and hence we obtain,

$$\int_{\Omega} \left[f(\nabla \bar{u}) + f(\nabla \bar{v}) - f\left(\frac{\nabla \bar{u} + \nabla \bar{v}}{2}\right)\right] = 0.$$

But since the integrand is nonnegative by convexity of f, this is only possible when

$$f(\nabla \bar{u}) + f(\nabla \bar{v}) - f\left(\frac{\nabla \bar{u} + \nabla \bar{v}}{2}\right) = 0 \quad \text{a.e.}$$

But this is impossible unless $\bar{u} = \bar{v}$ by the strict convexity of f. □

Of course, the simplest and also the prototype example for the last theorem is the p-Dirichlet integral.

$$\mathcal{D}_p[u] := \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p \, \mathrm{d}x.$$

The Euler-Lagrange equation is the p-Laplacian

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0.$$

The associated boundary value problem is

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \\ \quad u = u_0 & \text{on } \partial\Omega. \end{cases}$$

For $p = 2$, this is the Laplace equation as we have already seen. This partial differential operator is also **second order** and **elliptic**, but not **uniformly elliptic** if $p \neq 2$. It is **degenerate elliptic** for $2 < p < \infty$ and is **singular elliptic** for $1 < p < 2$.

1.3 Integrands with x dependence

The result we have established so far is a basic one, but it is too special to be of much use. As an example, suppose we want to solve the Dirichlet boundary value problem for the following PDE using variational method.

$$\operatorname{div}(A(x) \nabla u) = 0 \qquad \text{in } \Omega,$$

where A is a bounded and measurable, symmetric, uniformly positive-definite non-constant matrix-field. The associated energy functional is

$$I[u] := \frac{1}{2}\int_{\Omega} \langle A(x)\nabla u(x), \nabla u(x)\rangle \,\mathrm{d}x.$$

This does not fall into the category covered by our theorem since here f depends on x and ξ, and not just ξ.

1.3.1 Weak lower semicontinuity with x dependence

As we have already seen in the last example, we want to allow merely measurable dependece on x.

Definition 4 (Carathéodory functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$. f is called a Carathéodory function if*

- *$\zeta \mapsto f(x,\zeta)$ is continuous for a.e. $x \in \Omega$,*
- *$x \mapsto (x,\zeta)$ is measurable for every $\zeta \in \mathbb{R}^M$.*

Remark 5. *Roughly, $f = f(x,\xi)$ is a Carathéodory function when it depends measurably on x for every ξ and continuously on ξ for a.e. $x \in \Omega$.*

Similarly, $f = f(x,u,\xi)$ is a Carathéodory function when it depends measurably on x for every (u,ξ) and continuously on (u,ξ) for a.e. $x \in \Omega$.

Theorem 6 (sequential weak lower semicontinuity with x dependence)**.** *Let $n \geq 2, N \geq 1$ be integers and $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$, $f = f(x,\xi)$ be a Carathéodory function satisfying*

$$f(x,\xi) \geq \langle a(x), \xi\rangle + b(x)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{N\times n}$ for some $a \in L^{p'}\left(\Omega;\mathbb{R}^{N\times n}\right)$ and some $b \in L^1(\Omega)$. Let

$$I[u] := \int_{\Omega} f(x, \nabla u(x)) \,\mathrm{d}x.$$

Let $\xi \mapsto f(x,\xi)$ be convex for a.e. $x \in \Omega$ and

$$u_s \rightharpoonup u \quad \text{in } W^{1,p}\left(\Omega;\mathbb{R}^N\right).$$

Then we have

$$\liminf_{s\to\infty} I[u_s] \geq I[u].$$

The proof is exactly similar to the last weak lower semicontinuity theorem. The changes are just cosmetic. The moral of the story here is that if there is no explicit u dependence, even measurable dependence on x can be handled easily. This would change considerably when we shall deal with functions with both x and u dependence.

Existence of minimizer with x dependence

Theorem 7. *Let $n \geq 2, N \geq 1$ be integers, $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ be given. Let $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function satisfying*

$$f(x, \xi) \geq c\,|\xi|^p + b(x)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{N \times n}$ for some $c > 0$ and some $b \in L^1(\Omega)$. Assume $\xi \mapsto f(x, \xi)$ be convex for a.e. $x \in \Omega$. Let

$$I[u] := \int_\Omega f(\nabla u(x))\, \mathrm{d}x.$$

If $I[u_0] < \infty$, then the following problem

$$\inf\left\{I[u] : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)\right\} = m$$

admits a minimizer. If $\xi \mapsto f(x, \xi)$ is strictly convex for a.e. $x \in \Omega$, then the minimizer is unique.

The proof once again is exactly as before. The only thing that is different is how to show the inequality

$$f(x, \xi) \geq \langle a(x), \xi \rangle + b(x)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{N \times n}$ for some $a \in L^{p'}\left(\Omega; \mathbb{R}^{N \times n}\right)$ and some $b \in L^1(\Omega)$. By hypothesis, we have the inequality

$$f(x, \xi) \geq c\,|\xi|^p + b(x)$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{N \times n}$ for some $c > 0$ and some $b \in L^1(\Omega)$. But this implies the earlier inequality by taking $a \equiv 0$ with the same b.

1.4 Integrands with x and u dependence

Unfortunately, our hypotheses still leave out important problems. For example, the PDE

$$\Delta u = f \qquad \text{in } \Omega$$

where $f \not\equiv 0$. Indeed, the energy functional is

$$I[u] = \int_\Omega \left[\frac{1}{2}\left|\nabla u(x)\right|^2 + \langle f(x), u(x) \rangle\right] \mathrm{d}x.$$

This depends not only on x, but also explicitly on u. However, here at least the dependence on u is linear. The functional

$$I[u] = \int_\Omega \left[\frac{1}{2}\left|\nabla u(x)\right|^2 + \frac{\lambda}{2}\left|u(x)\right|^2\right] \mathrm{d}x,$$

which corresponds to the eigenvalue problem

$$\Delta u = \lambda u \qquad \text{in } \Omega$$

is a more general and an important example.

Bibliography

[1] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.