Introduction to the Calculus of Variations
Lecture Notes
Lecture 16

Swarnendu Sil

Spring 2021, IISc

Chapter 1

Sobolev spaces

1.1 Definitions

1.2 Elementary properties

1.3 Approximation and extension

1.4 Traces

1.5 Sobolev inequalities and Sobolev embeddings

1.5.1 Gagliardo-Nirenberg-Sobolev inequalities

1.5.2 Poincaré-Sobolev inequalities

1.5.3 Morrey's inequality

1.5.4 Rellich-Kondrachov compact embeddings

Compactness in L^p spaces

Now we proceed to the question of compactness of the Sobolev embeddings. But before stating the result, we first record a criterion for compactness in $L^q(\Omega)$.

Theorem 1 (Kolmogorov-M.Riesz-Frechet)**.** *Let $\mathcal{F}$ be a bounded subset of $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$ such that*

$$\lim_{|h| \to 0} \|\tau_h u - u\|_{L^p(\mathbb{R}^n)} = 0 \qquad \text{uniformly in } u \in \mathcal{F}.$$

*Then the closure of $\mathcal{F}|_\Omega$ is **compact** in $L^p(\Omega)$ for any measurable $\Omega \subset \mathbb{R}^n$ with finite measure.*

Remark 2. *Here τ_h is the translation operator, i.e.*

$$\tau_h u(x) := u(x+h) \qquad \text{for all } x \in \mathbb{R}^n.$$

Since this result is often proved in measure and integral courses while studying L^p spaces, we omit the proof. A detailed proof can be found at page 111 in Theorem 4.26 of [1].

Rellich-Kondrachov compact embeddings

Now we state our main result.

Theorem 3 (Rellich-Kondrachov compact embeddings)**.** *Let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Then the following injections are all* ***compact***

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } 1 \leq q < p^* && \text{for } 1 \leq p < n, \\ &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } 1 \leq q < \infty && \text{for } p = n, \\ &W^{1,p}(\Omega) \hookrightarrow C\left(\overline{\Omega}\right) && && \text{for } n < p < \infty. \end{aligned}$$

Proof. The case $p > n$ follows from Morrey's inequality and Ascoli-Arzela theorem. The case $p = n$ can be deduced from the case $1 \leq p < n$ case. So we just prove this later case. Also, since Ω is bounded, clearly it is enough to prove the result for $p \leq q < p^*$.

By using extension and the Kolmogororv-M.Riesz-Frechet theorem, we only need to show

$$\lim_{|h| \to 0} \|\tau_h u - u\|_{L^q(\mathbb{R}^n)} = 0 \qquad \text{uniformly in } u \in \mathcal{F},$$

for any $p \leq q < p^*$ and any bounded subset $\mathcal{F} \subset W^{1,p}(\mathbb{R}^n)$. But we have, by interpolation,

$$\begin{aligned} \|\tau_h u - u\|_{L^q} &\leq \|\tau_h u - u\|_{L^p}^{\alpha} \|\tau_h u - u\|_{L^{p^*}}^{1-\alpha} \\ &\leq c\,|h|^{\alpha} \|\nabla u\|_{L^p}^{\alpha} \|u\|_{L^{p^*}}^{1-\alpha} \\ &\leq cM\,|h|^{\alpha}. \end{aligned}$$

This proves the theorem. □

Remark 4.

- *In fact, using interpolation inequalities, as we did in the proof above, but for Hölder spaces instead of L^p spaces, we can actually conclude that if $n < \infty$, $W^{1,p}(\Omega)$ embeds compactly into $C^{0,\alpha}\left(\overline{\Omega}\right)$ for every $0 \leq \alpha < 1 - \frac{n}{p}$.*

- *Note that the theorem* ***does not claim*** *that the embedding of $W^{1,p}$ into L^{p^*} in the case $1 \leq p < n$ is compact. In fact, this injection, though continuous, is never compact. See Example 5 for a generic counterexample. The same counter-example also shows that the continuous injection of $W^{1,p}$ into $C^{0,1-\frac{n}{p}}$ is also never compact for $n < p < \infty$. It also shows that $W^{1,n} \cap L^{\infty}$ also does not compactly embed in L^{∞}.*

Lack of compactness at the critical exponent

Example 5. *Let* $u \in C_c^{\infty}(B_1^n)$ *with* $u(0) \neq 0$. *Set*

$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad v_\varepsilon(x) := \left(\frac{1}{\varepsilon}\right)^{\frac{n-p}{p}} u_\varepsilon(x).$$

Now $u_\varepsilon, v_\varepsilon \in W^{1,p}(B_1^n)$ *for every* $\varepsilon > 0$.

Case I: the case $1 \leq p < n$,

We compute

$$\|v_\varepsilon\|_{L^q(B_1^n)} = \left(\frac{1}{\varepsilon}\right)^{\frac{n-p}{p}} \|u_\varepsilon\|_{L^q(B_1^n)} = \varepsilon^{n\left(\frac{1}{q}-\frac{1}{p}+\frac{1}{n}\right)} \|u\|_{L^q(B_1^n)}.$$

for any $1 \leq q \leq p^*$. *Similarly, we have*

$$\|\nabla v_\varepsilon\|_{L^p(B_1^n)} = \|\nabla u\|_{L^p(B_1^n)}.$$

Thus, the sequence $\{v_\varepsilon\}_\varepsilon$ *is uniformly bounded in* $W^{1,p}(B_1^n)$ *and* $v_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *in* $L^q(B_1^n)$ *for any* $1 \leq q < p^*$. *Thus if* $\{v_\varepsilon\}_\varepsilon$ *admits a convergent subsequence, by uniqueness of the limit, the limit must be the zero function. But no subsequence of* $\{v_\varepsilon\}_\varepsilon$ *can converge strongly in* L^{p^*} *to the zero function, as we clearly have*

$$\|v_\varepsilon\|_{L^{p^*}(B_1^n)} = \|u\|_{L^{p^*}(B_1^n)} > 0 \qquad \text{for every } \varepsilon > 0.$$

So there can be no subsequence which is strongly convergent in $L^{p^*}(B_1^n)$ *at all.*

Case II: the case $n < p < \infty$,

We can prove exactly as before that $\{v_\varepsilon\}_\varepsilon$ *is uniformly bounded in* $W^{1,p}(B_1^n)$. *Since* $p > n$, *it is also clear that*

$$\|v_\varepsilon\|_{C^0(\overline{B_1^n})} \leq \varepsilon^{\frac{p-n}{p}} \|u\|_{C^0(\overline{B_1^n})} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

So as before, if subsequential limit must be the zero function. Now it is clear that $\operatorname{supp} v_\varepsilon \subset B(0,\varepsilon)$. *Now, for any* $\varepsilon > 0$ *and for any* $y \in \partial B(0,\varepsilon)$, *we compute*

$$\frac{|v_\varepsilon(0) - v_\varepsilon(y)|}{|y-0|^{\left(1-\frac{n}{p}\right)}} = \frac{|v_\varepsilon(0)|}{|y|^{\left(1-\frac{n}{p}\right)}} = \frac{\varepsilon^{\frac{p-n}{p}} |u(0)|}{\varepsilon^{\left(1-\frac{n}{p}\right)}} = |u(0)| > 0.$$

Thus, we deduce

$$[v_\varepsilon]_{C^{0,1-\frac{n}{p}}(\overline{B_1^n})} \geq \frac{|v_\varepsilon(0) - v_\varepsilon(y)|}{|y-0|^{\left(1-\frac{n}{p}\right)}} = |u(0)| > 0,$$

for any $\varepsilon > 0$. *Thus, there can be no subsequence of* $\{v_\varepsilon\}_\varepsilon$ *which is strongly convergent in* $C^{0,1-\frac{n}{p}}\left(\overline{B_1^n}\right)$ *at all.*

Case III: the case $p = n$***,***

Note that for $p = n$, $v_\varepsilon = u_\varepsilon$. *In general,* $W^{1,n}$ *functions need not be* L^∞. *But here, since* u *being smooth is* L^∞, *so is* v_ε *for every* $\varepsilon > 0$. *Also, we have, for every* $\varepsilon > 0$,

$$\|v_\varepsilon\|_{L^\infty(B_1^n)} = \|u\|_{L^\infty(B_1^n)}.$$

As before, $\{v_\varepsilon\}_\varepsilon$ *is uniformly bounded in* $W^{1,n}(B_1^n)$ *and hence by virtue of the last equality, also in* $W^{1,n}(B_1^n) \cap L^\infty(B_1^n)$. *But the same equality tells us that no subsequence can converge to zero function in the* L^∞, *which is the only possible candidate for the limit as before. This proves that the embedding of* $W^{1,n}(B_1^n) \cap L^\infty(B_1^n)$ *into* $L^\infty(B_1^n)$. *can not be compact.*

Summary of Sobolev embeddings

Let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth.

- If $1 \le p < n$, then the injections

  $$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } 1 \le q \le p^*$$

  are **continuous**. These injections are **compact** for $1 \le q < p^*$, but **not** for $q = p^*$.

- The injections

  $$W^{1,n}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } 1 \le q < \infty$$

  are all **continuous** and **compact**. $W^{1,n}(\Omega)$ **does not** embed continuously in $L^\infty(\Omega)$. Also, the injection of $W^{1,n}(\Omega) \cap L^\infty(\Omega)$ into $L^\infty(\Omega)$ is **not** compact.

- If $n < p < \infty$, then the injections

  $$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right) \qquad \text{for all } 0 \le \alpha \le 1 - \frac{n}{p}$$

  are **continuous**. These injections are **compact** for $0 \le \alpha < 1 - \frac{n}{p}$, but **not** for $\alpha = 1 - \frac{n}{p}$.

Chapter 2

Direct methods

2.1 Dirichlet Integral

Now we are ready to begin our study of the modern direct methods in the calculus of variations. Let $n \geq 2, N \geq 1$ be integers and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$. Then the functional

$$\mathcal{D}\left[u\right] := \frac{1}{2}\int_{\Omega} \left|\nabla u\right|^2$$

is called the **Dirichlet integral** of u. Note that for any $u \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$, we have

$$\mathcal{D}\left[u\right] < \infty.$$

Now we want to minimize the Dirichlet integral with a given Dirichlet boundary value.

Theorem 6. *Let $n \geq 2, N \geq 1$ be integers and let $\Omega \subset \mathbb{R}^n$ be open bounded and smooth. Let $u_0 \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ be given. Then the following problem*

$$\inf\left\{\mathcal{D}\left[u\right] := \frac{1}{2}\int_{\Omega} \left|\nabla u\right|^2 : u \in u_0 + W_0^{1,2}\left(\Omega; \mathbb{R}^N\right)\right\} = m$$

admits an ***unique*** *minimizer $\bar{u} \in u_0 + W_0^{1,2}\left(\Omega; \mathbb{R}^N\right)$. Moreover, $\bar{u}$ is a* ***weak solution*** *of the Dirichlet boundary value problem*

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \Omega, \\ \bar{u} = u_0 & \text{on } \partial\Omega. \end{cases}$$

i.e. satisfies the weak form of the Euler-Lagrange equation

$$\int_{\Omega} \langle \nabla \bar{u}, \nabla \phi \rangle = 0 \qquad \text{for all } \phi \in W_0^{1,2}\left(\Omega; \mathbb{R}^N\right).$$

Proof. Let $\{u_s\}_{s\geq 1} \subset u_0 + W_0^{1,2}\left(\Omega; \mathbb{R}^N\right)$ be a minimizing sequence, i.e

$$\mathcal{D}\left[u_s\right] \to m \qquad \text{as } s \to \infty.$$

Uniform bound for minimizing sequence

Since $u_s - u_0 \in W_0^{1,2}\left(\Omega; \mathbb{R}^N\right)$ for every $s \geq 1$, by Poincaré inequality, we have

$$\begin{aligned}\left\|u_s - u_0\right\|_{W^{1,2}} &\leq c \left\|\nabla u_s - \nabla u_0\right\|_{L^2} \\ &\leq c\sqrt{\mathcal{D}\left[u_s\right]} + c \left\|\nabla u_0\right\|_{L^2} \\ &\leq c\sqrt{m+1} + c \left\|\nabla u_0\right\|_{L^2}.\end{aligned}$$

Thus, we have

$$\left\|u_s\right\|_{W^{1,2}} \leq \left\|u_s - u_0\right\|_{W^{1,2}} + \left\|u_0\right\|_{W^{1,2}} \leq c\sqrt{m+1} + c \left\|u_0\right\|_{W^{1,2}}.$$

This proves that $\{u_s\}_{s\geq 1}$ is uniformly bounded in $W^{1,2}\left(\Omega; \mathbb{R}^N\right)$.

Since $W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ is reflexive, the uniform bound implies that there exists $\bar{u} \in W^{1,2}\left(\Omega; \mathbb{R}^N\right)$ such that up to the extraction of a subsequence, which we do not relabel, we have

$$u_s \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,2}\left(\Omega; \mathbb{R}^N\right).$$

Sequential weak lower semicontinuity

Now we wish to prove that

$$\liminf_{s\to\infty} \mathcal{D}\left[u_s\right] \geq \mathcal{D}\left[\bar{u}\right].$$

We have,

$$\begin{aligned}2\mathcal{D}\left[u_s\right] &= \int_\Omega \langle \nabla u_s - \nabla\bar{u} + \nabla\bar{u}, \nabla u_s - \nabla\bar{u} + \nabla\bar{u}\rangle \\ &= \int_\Omega \langle \nabla u_s - \nabla\bar{u}, \nabla u_s - \nabla\bar{u}\rangle + 2\int_\Omega \langle \nabla u_s - \nabla\bar{u}, \nabla\bar{u}\rangle \\ &\qquad\qquad + \int_\Omega \langle \nabla\bar{u}, \nabla\bar{u}\rangle \\ &\geq 2\mathcal{D}\left[\bar{u}\right] + 2\int_\Omega \langle \nabla u_s - \nabla\bar{u}, \nabla\bar{u}\rangle.\end{aligned}$$

Since

$$u_s \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,2}\left(\Omega; \mathbb{R}^N\right).$$

implies

$$\nabla u_s \rightharpoonup \nabla\bar{u} \qquad \text{weakly in } L^2\left(\Omega; \mathbb{R}^N\right),$$

we deduce that

$$\lim_{s\to\infty}\int_\Omega \langle \nabla u_s - \nabla \bar{u}, \nabla \bar{u}\rangle = 0.$$

This proves the weak lower semicontinuity.

Thus, we have

$$m \leq \mathcal{D}\left[\bar{u}\right] \leq \liminf_{s\to\infty} \mathcal{D}\left[u_s\right] = m.$$

Hence $\bar{u}$ is a minimzer.

Uniqueness

Suppose $\bar{u}$ and $\bar{v}$ are both minimizers. Then let $\bar{w} := \frac{1}{2}\left(\bar{u}+\bar{v}\right).$ Then we can see

$$m \leq \mathcal{D}\left[\bar{w}\right] \leq \frac{1}{2}\mathcal{D}\left[\bar{u}\right] + \frac{1}{2}\mathcal{D}\left[\bar{v}\right] \leq m.$$

So $\bar{w}$ is also a minimizer and hence we obtain,

$$\int_\Omega \left(\frac{1}{2}\left|\nabla \bar{u}\right|^2 + \frac{1}{2}\left|\nabla \bar{u}\right|^2 - \left|\frac{\nabla \bar{u} + \nabla \bar{v}}{2}\right|^2\right) = 0.$$

But this implies

$$\frac{1}{2}\left|\nabla \bar{u}\right|^2 + \frac{1}{2}\left|\nabla \bar{u}\right|^2 - \left|\frac{\nabla \bar{u} + \nabla \bar{v}}{2}\right|^2 = 0 \quad \text{a.e.}$$

But this is impossible unless $u = v$ by the strict convexity of the function $\xi \mapsto \left|\xi\right|^2.$

Euler-Lagrange equations

Now if $\bar{u}$ is a minimizer, we must have

$$\frac{d}{dt}\left(\mathcal{D}\left[\bar{u}+t\phi\right]\right)\Big|_{t=0} = 0$$

for any $\phi \in C_c^\infty\left(\Omega;\mathbb{R}^N\right).$ Thus we compute

$$\begin{aligned} 0 &= \lim_{t\to 0}\frac{1}{2t}\int_\Omega \left[\left|\nabla \bar{u} + t\nabla\phi\right|^2 - \left|\nabla \bar{u}\right|^2\right] \\ &= \lim_{t\to 0}\frac{1}{2t}\int_\Omega \left[t\left\langle \nabla\phi, \nabla\bar{u}\right\rangle + t^2\left|\phi\right|^2\right] \\ &= \int_\Omega \left\langle \nabla\phi, \nabla\bar{u}\right\rangle. \end{aligned}$$

But the fact that $\nabla\bar{u} \in L^2$ and the density of C_c^∞ functions in $W_0^{1,2}$ implies that the identity holds for any $\phi \in W_0^{1,2}$ as well, i.e.

$$\int_\Omega \left\langle \nabla\phi, \nabla\bar{u}\right\rangle = 0 \qquad \text{for any } \phi \in W_0^{1,2}\left(\Omega;\mathbb{R}^N\right).$$

This completes the proof. □

Bibliography

[1] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.