Introduction to the Calculus of Variations
Lecture Notes
Lecture 15

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Chapter 1

Sobolev spaces

1.1 Definitions

1.2 Elementary properties

1.3 Approximation and extension

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1.5 Sobolev inequalities and Sobolev embeddings

1.5.1 Gagliardo-Nirenberg-Sobolev inequalities

1.5.2 Poincaré-Sobolev inequalities

Poincaré inequality on balls

Now we plan to derive a local version of a Poincaré inequality.

Lemma 1 (Local Poincaré inequality)**.** *For every $1 \leq p < \infty$, there exists a constant $c > 0$, depending only on n and p such that*

$$\int_{B(x,r)} |u(y) - u(z)|^p \ \mathrm{d}y \leq c r^{n+p-1} \int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y-z|^{n-1}} \ \mathrm{d}y, \tag{1.1}$$

for every ball $B(x,r) \subset \mathbb{R}^n$, every $z \in B(x,r)$ and every $u \in W^{1,p}(\mathbb{R}^n)$.

Remark 2. *Note that like the Poincaré inequality, here also the estimate controls certain integral related to u by integrals related to ∇u.*

Proof. We can obviously assume $u \in C^1(\mathbb{R}^n)$. For $y, z \in B(x,r)$, we have,

$$u(y) - u(z) = \int_0^1 \frac{d}{dt} u(z + t(y-z)) \, \mathrm{d}t$$
$$= \int_0^1 \langle \nabla u(z + t(y-z)), y - z \rangle \, \mathrm{d}t$$

Thus, we have,

$$|u(y) - u(z)|^p \leq |y - z|^p \int_0^1 |\nabla u(z + t(y-z))|^p \, \mathrm{d}t. \tag{1.2}$$

Let $k > 0$ be a number such that $B(x,r) \subset B(z,kr)$ for any $z \in B(x,r)$. Now we plan to integrate this over $y \in \partial B(z,s)$ for any $s > 0$ and then integrate w.r.t. s from 0 to kr.

Now, for any $s > 0$, integrating (1.2) over $y \in \partial B(z,s)$, we have,

$$\int_{B(x,r) ap \partial B(z,s)} |u(y) - u(z)|^p \, \mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\leq s^p \int_0^1 \int_{B(x,r) \cap \partial B(z,s)} |\nabla u(z + t(y-z))|^p \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t.$$

Putting $w = z + t(y - z)$ and changing variables, this implies,

$$\int_{B(x,r) \cap \partial B(z,s)} |u(y) - u(z)|^p \, \mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\leq s^p \int_0^1 \frac{1}{t^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^p \, \mathrm{d}\mathcal{H}^{n-1}(w) \, \mathrm{d}t$$
$$= s^{n+p-1} \int_0^1 \frac{1}{(ts)^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^p \, \mathrm{d}\mathcal{H}^{n-1}(w) \, \mathrm{d}t.$$

The RHS of the last inequality is

$$s^{n+p-1} \int_0^1 \frac{1}{(ts)^{n-1}} \int_{B(x,r) \cap \partial B(z,ts)} |\nabla u(w)|^p \, \mathrm{d}\mathcal{H}^{n-1}(w) \, \mathrm{d}t$$
$$= s^{n+p-1} \int_0^1 \int_{B(x,r) \cap \partial B(z,ts)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} \, \mathrm{d}\mathcal{H}^{n-1}(w) \, \mathrm{d}t$$
$$= s^{n+p-2} \int_0^s \int_{B(x,r) \cap \partial B(z,\theta)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} \, \mathrm{d}\mathcal{H}^{n-1}(w) \, \mathrm{d}\theta$$
$$= s^{n+p-2} \int_{B(x,r) \cap B(z,s)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} \, \mathrm{d}w$$
$$\leq s^{n+p-2} \int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w - z|^{n-1}} \, \mathrm{d}w.$$

So we arrive at

$$\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p \ \mathrm{d}\mathcal{H}^{n-1}(y) \\ \leq s^{n+p-2}\int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} \ \mathrm{d}w.$$

Integrating w.r.t s from 0 to kr and noticing that $B(x,r)\subset B(z,kr)$, we deduce

$$\begin{aligned}\int_{B(x,r)} |u(y)-u(z)|^p \ \mathrm{d}y &\leq \int_{B(x,r)\cap B(z,kr)} |u(y)-u(z)|^p \ \mathrm{d}y \\ &= \int_0^{kr}\int_{B(x,r)\cap\partial B(z,s)} |u(y)-u(z)|^p \ \mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}s \\ &\leq \int_0^{kr} s^{n+p-2}\mathrm{d}s\int_{B(x,r)} \frac{|\nabla u(w)|^p}{|w-z|^{n-1}} \ \mathrm{d}w \\ &\leq cr^{n+p-1}\int_{B(x,r)} \frac{|\nabla u(y)|^p}{|y-z|^{n-1}} \ \mathrm{d}y.\end{aligned}$$

This proves the lemma. □

Poincaré inequality with mean on balls

We now prove a Poincaré type inequality for $W^{1,p}$ functions.

Theorem 3 (Poincaré inequality with mean on balls)**.** *For every* $1\leq p<\infty$, *there exists a constant* $c>0$, *depending only on* n *and* p *such that*

$$⨍_{B(x,r)} \left|u(y)-(u)_{B(x,r)}\right|^p \ \mathrm{d}y \leq cr^p ⨍_{B(x,r)} |\nabla u(y)|^p \ \mathrm{d}y, \tag{1.3}$$

for every ball $B(x,r)\subset\mathbb{R}^n$ *and every* $u\in W^{1,p}(\mathbb{R}^n)$.

Remark 4. *Here the integral mean is*

$$(u)_{B(x,r)} := \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y) \ \mathrm{d}y$$

and the notation for averaged integral is defined as

$$⨍_{B(x,r)} f(y) \ \mathrm{d}y = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y) \ \mathrm{d}y.$$

Proof. As usual we can assume $u \in C^1(\mathbb{R}^n)$. Now we have,

$$\begin{aligned}
&\fint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\
&\qquad = \fint_{B(x,r)} \left| \fint_{B(x,r)} (u(y) - u(z)) \, \mathrm{d}z \right|^p \, \mathrm{d}y \\
&\qquad \leq \fint_{B(x,r)} \fint_{B(x,r)} |u(y) - u(z)|^p \, \mathrm{d}y\mathrm{d}z
\end{aligned}$$

Now, applying Lemma 1 to estimate the RHS, we deduce

$$\begin{aligned}
&\fint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\
&\qquad \leq c \fint_{B(x,r)} r^{p-1} \int_{B(x,r)} \frac{|\nabla u(z)|^p}{|y-z|^{n-1}} \, \mathrm{d}z\mathrm{d}y \\
&\qquad \leq c r^{p-1} \fint_{B(x,r)} \int_{B(x,r)} \frac{|\nabla u(z)|^p}{|y-z|^{n-1}} \, \mathrm{d}z\mathrm{d}y
\end{aligned}$$

Now using Fubini, we deduce

$$\begin{aligned}
&\fint_{B(x,r)} \left| u(y) - (u)_{B(x,r)} \right|^p \, \mathrm{d}y \\
&\qquad \leq c r^{p-1} \fint_{B(x,r)} \int_{B(x,r)} \frac{|\nabla u(z)|^p}{|y-z|^{n-1}} \, \mathrm{d}z\mathrm{d}y \\
&\qquad = c r^{p-1} \int_{B(x,r)} |\nabla u(z)|^p \left(\fint_{B(x,r)} \frac{1}{|y-z|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\
&\qquad \leq c r^{p-1} \int_{B(x,r)} |\nabla u(z)|^p \left(\frac{1}{r^n} \int_{B(z,kr)} \frac{1}{|y-z|^{n-1}} \, \mathrm{d}y \right) \mathrm{d}z \\
&\qquad = c \frac{r^p}{r^n} \int_{B(x,r)} |\nabla u(z)|^p \, \mathrm{d}z \\
&\qquad = c r^p \fint_{B(x,r)} |\nabla u(z)|^p \, \mathrm{d}z.
\end{aligned}$$

□

Poincaré-Sobolev inequality with mean on balls

As a corollary, we derive the Poincaré-Sobolev inequality with mean on balls.

Theorem 5 (Poincaré-Sobolev inequality with mean on balls)**.** *For every* $1 \leq$

$p < n$, there exists a constant $c > 0$, depending only on n and p such that

$$\left(\fint_{B(x,r)} \left|u(y) - (u)_{B(x,r)}\right|^{p^*} \mathrm{d}y\right)^{\frac{1}{p^*}} \leq cr \left(\fint_{B(x,r)} |\nabla u(y)|^p \mathrm{d}y\right)^{\frac{1}{p}}, \tag{1.4}$$

for every ball $B(x,r) \subset \mathbb{R}^n$ and every $u \in W^{1,p}(\mathbb{R}^n)$.

Proof. We first prove the inequality

$$\left(\fint_{B(x,r)} |v(y)|^{p^*} \mathrm{d}y\right)^{\frac{1}{p^*}} \leq c\left(r^p \fint_{B(x,r)} |\nabla v(y)|^p \mathrm{d}y + \fint_{B(x,r)} |v(y)|^p \mathrm{d}y\right)^{\frac{1}{p}},$$

for every ball $B(x,r) \subset \mathbb{R}^n$ and for every $v \in W^{1,p}(\mathbb{R}^n)$ with $1 \leq p < n$.

Note that replacing v by $\frac{1}{r}v(ry)$ and translation, we can assume that $x = 0$ and $r = 1$. But in this case, the inequality above is just the Poincaré-Sobolev inequality for the bounded domain $B(0,1) \subset \mathbb{R}^n$.

This proves the inequality.

Now we apply this inequality to the function $v := u - (u)_{B(x,r)}$. We obtain

$$\left(\fint_{B(x,r)} \left|u - (u)_{B(x,r)}\right|^{p^*}\right)^{\frac{1}{p^*}} \leq c\left(r^p \fint_{B(x,r)} |\nabla u|^p + \fint_{B(x,r)} \left|u - (u)_{B(x,r)}\right|^p\right)^{\frac{1}{p}}.$$

Now we use the Poincaré inequality with mean on balls to estimate the last term to obtain

$$\left(\fint_{B(x,r)} \left|u - (u)_{B(x,r)}\right|^{p^*}\right)^{\frac{1}{p^*}} \leq c\left(r^p \fint_{B(x,r)} |\nabla u|^p\right)^{\frac{1}{p}}.$$

This proves the theorem. □

1.5.3 Morrey's inequality

Now we prove an important inequality.

Theorem 6 (Morrey's inequality)**.** *For every $n < p < \infty$, there exists a constant $c > 0$, depending only on n and p such that*

$$|u(y) - u(z)| \leq cr \left(\fint_{B(x,r)} |\nabla u(y)|^p \mathrm{d}y\right)^{\frac{1}{p}}, \tag{1.5}$$

for a.e. $y, z \in B(x,r)$ for every ball $B(x,r) \subset \mathbb{R}^n$ and for every $u \in W^{1,p}(\mathbb{R}^n)$.

Proof. We use the local Poincaré inequality lemma with $p = 1$ to deduce

$$
\begin{aligned}
&|u(y) - u(z)| \\
&\leq ⨍_{B(x,r)} (|u(y) - u(w)| + |u(w) - u(z)|) \, \mathrm{d}w \\
&\leq c \int_{B(x,r)} |\nabla u(w)| \left(|y - w|^{1-n} + |z - w|^{1-n} \right) \, \mathrm{d}w \\
&\overset{\text{Hölder}}{\leq} c \left(\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \right)^{\frac{p-1}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}.
\end{aligned} \tag{1.6}
$$

Now, since $p^{'} = \frac{p}{p-1} > 1$, the function $t \mapsto t^{\frac{p}{p-1}}$ is convex. Hence we deduce,

$$
\left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}} \left(|y - w|^{\frac{p(1-n)}{p-1}} + |z - w|^{\frac{p(1-n)}{p-1}} \right).
$$

Using this, we obtain

$$
\begin{aligned}
&\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \\
&\qquad \leq c \left(\int_{B(x,r)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w + \int_{B(x,r)} |z - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w \right).
\end{aligned} \tag{1.7}
$$

Now, as before, for any $y \in B(x, r)$, we can find $k > 0$ such that $B(x, r) \subset B(y, kr)$. Thus, we can estimate

$$
\begin{aligned}
\int_{B(x,r)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w &\leq \int_{B(y,kr)} |y - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w \\
&= \int_0^{kr} \int_{\mathbb{S}^{n-1}} \rho^{\frac{p(1-n)}{p-1}} \cdot \rho^{n-1} \, \mathrm{d}\rho \, \mathrm{d}\theta \\
&= \int_0^{kr} \int_{\mathbb{S}^{n-1}} \rho^{(n-1)\left(1 - \frac{p}{p-1}\right)} \, \mathrm{d}\rho \, \mathrm{d}\theta = c r^{\frac{p-n}{p-1}}.
\end{aligned}
$$

Similarly, we can also establish the estimate

$$
\int_{B(x,r)} |z - w|^{\frac{p(1-n)}{p-1}} \, \mathrm{d}w \leq c r^{\frac{p-n}{p-1}}.
$$

Combining these last two estimates with (1.7), we obtain

$$
\int_{B(x,r)} \left(|y - w|^{1-n} + |z - w|^{1-n} \right)^{\frac{p}{p-1}} \, \mathrm{d}w \leq c r^{\frac{p-n}{p-1}}.
$$

Plugging this estimate into (1.6), we arrive at

$$
|u(y) - u(z)| \leq c r^{1 - \frac{n}{p}} \left(\int_{B(x,r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}.
$$

This proves the inequality. $\square$

Sobolev embedding for $p > n$

Morrey's inequality implies that $W^{1,p}$ functions with $p > n$ are Hölder continuous with exponent $\alpha = 1 - \frac{n}{p}$.

Theorem 7 (Sobolev embedding in $\mathbb{R}^n$ for $p > n$)**.** *Let $n < p < \infty$. Then $W^{1,p}(\mathbb{R}^n)$ continuously embeds into $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$.*

Proof. By Morrey's inequality, for a.e. $x, y \in \mathbb{R}^n$ with $|x - y| = r$, we have,

$$\begin{aligned} |u(x) - u(y)| &\leq c r^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} \\ &= c\,|x-y|^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}} \\ &\leq c\,|x-y|^{1-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, for a.e. $x, y \in \mathbb{R}^n$, we have

$$\frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq c \left(\int_{\mathbb{R}^n} |\nabla u(w)|^p \, \mathrm{d}w \right)^{\frac{1}{p}}. \tag{1.8}$$

This implies that u agrees a.e. with a continuous function $\tilde{u}$. Indeed, let $A \subset \mathbb{R}^n$ be the subset of measure zero such that (1.8) holds for any $x, y \in \mathbb{R}^n \setminus A$. Then $u|_{\mathbb{R}^n \setminus A}$ is continuous and since $\mathbb{R}^n \setminus A$ is dense in $\mathbb{R}^n$, there exists a unique continuous extension $\tilde{u}$ such that (1.8) holds for all $x, y \in \mathbb{R}^n$ for $\tilde{u}$. From now on, by a harmless abuse of notation, we simply denote this extension by u itself. Thus, taking supremum as x, y varies in $\mathbb{R}^n$, we have,

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} := \sup_{x,y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq c\,\|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{1.9}$$

This estimates the Hölder seminorm. Now all that remains is to estimate the C^0 norm. For any $x \in \mathbb{R}^n$, using the local Poincaré inequality lemma with $p = 1$, we have

$$
\begin{aligned}
|u(x)| &\leq \fint_{B(x,1)} |u(x)| \ \mathrm{d}y \\
&\leq \fint_{B(x,1)} |u(x) - u(y)| \ \mathrm{d}y + \fint_{B(x,1)} |u(y)| \ \mathrm{d}y \\
&\leq \int_{B(x,1)} |\nabla u(y)| \, |x-y|^{1-n} \ \mathrm{d}y + \fint_{B(x,1)} |u(y)| \ \mathrm{d}y \\
&\overset{\text{Hölder}}{\leq} \left(\int_{B(x,1)} |\nabla u(y)|^p \ \mathrm{d}y \right)^{\frac{1}{p}} \left(\int_{B(x,1)} \frac{1}{|x-y|^{\frac{(n-1)p}{p-1}}} \ \mathrm{d}y \right)^{\frac{p-1}{p}} \\
&\qquad\qquad + c \left(\int_{B(x,1)} |u(y)|^p \ \mathrm{d}y \right)^{\frac{1}{p}} \\
&\leq c \left(\int_{B(x,1)} |\nabla u(y)|^p \ \mathrm{d}y \right)^{\frac{1}{p}} + c \left(\int_{B(x,1)} |u(y)|^p \ \mathrm{d}y \right)^{\frac{1}{p}} \leq c \, \|u\|_{W^{1,p}(\mathbb{R}^n)} .
\end{aligned}
$$

Taking supremum as $x \in \mathbb{R}^n$, we have

$$
\|u\|_{C^0(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |u(x)| \leq c \, \|u\|_{W^{1,p}(\mathbb{R}^n)} .
$$

Thus, we obtain

$$
\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} := \|u\|_{C^0(\mathbb{R}^n)} + [u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq c \, \|u\|_{W^{1,p}(\mathbb{R}^n)} .
$$

This proves the result. □

As usual, the result for $\mathbb{R}^n$ implies, by extension, the result for bounded domains.

Theorem 8 (Sobolev embedding in bounded domains for $p > n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $n < p < \infty$. Then $W^{1,p}(\Omega)$ continuously embeds into $C^{0,\alpha}\left(\overline{\Omega}\right)$ for every $0 \leq \alpha \leq 1 - \frac{n}{p}$.*

$W^{1,\infty}$ and Lipschitz functions

As a consequence, we can deduce

Theorem 9 ($W^{1,\infty} = C^{0,1}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. Then*

$$
W^{1,\infty}(\Omega) = C^{0,1}\left(\overline{\Omega}\right) \qquad \textit{(with equivalent norms).}
$$

Proof. Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for any $n < p < \infty$, by the last theorem, for any $x, y \in \overline{\Omega}$, we obtain

$$
|u(x) - u(y)| \leq c \, |x-y|^{1-\frac{n}{p}} \, \|u\|_{W^{1,p}(\Omega)} .
$$

Letting $p \to \infty$ and noting that

$$\lim_{p\to\infty} \|u\|_{W^{1,p}(\Omega)} = \|u\|_{W^{1,\infty}(\Omega)},$$

we obtain the inequality

$$|u(x) - u(y)| \leq c\,|x-y|\,\|u\|_{W^{1,\infty}(\Omega)}.$$

But this implies

$$[u]_{C^{0,1}(\overline{\Omega})} := \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|} \leq c\,\|u\|_{W^{1,\infty}(\Omega)}.$$

Thus, we have established the continuous embedding

$$W^{1,\infty}(\Omega) \subset C^{0,1}\left(\overline{\Omega}\right).$$

The other inclusion is easy and was proved earlier in this chapter. This completes the proof. □