Introduction to the Calculus of Variations
Lecture Notes
Lecture 14

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Chapter 1

Sobolev spaces

1.1 Definitions

1.2 Elementary properties

1.3 Approximation and extension

1.4 Traces

1.5 Sobolev inequalities and Sobolev embeddings

1.5.1 Gagliardo-Nirenberg-Sobolev inequalities

Theorem 1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $1 \leq p < n$. Then there exists a constant $c > 0$, depending only on n and p such that we have the estimate*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left(\int_{\mathbb{R}^n} |\nabla u|^p \right)^{\frac{1}{p}} \tag{1.1}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$.

To prove this inequality, we need a simple lemma.

Lemma 2. *Let $n \geq 2$ and let $f_1, \ldots, f_n \in L^{n-1}\left(\mathbb{R}^{n-1}\right)$. For $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, set*

$$\hat{x}_i = (x_1, \ldots, \widehat{x_i}, \ldots, x_n) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Then the function

$$f(x) := \prod_{i=1}^{n} f_i(\hat{x}_i) \qquad \text{for } x \in \mathbb{R}^n$$

is in $L^1(\mathbb{R}^n)$ and we have the estimate

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof. $n = 2$ is just Fubini with equality in fact. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^2} |f| \, \mathrm{d}x_1 \mathrm{d}x_2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_1(x_2)| \, |f_2(x_1)| \, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \left(\int_{-\infty}^{\infty} |f_2(x_1)| \, \mathrm{d}x_1 \right) \left(\int_{-\infty}^{\infty} |f_1(x_2)| \, \mathrm{d}x_2 \right).\end{aligned}$$

Now to prove by induction, we assume the result holds for some $n \geq 2$ and show it for $n+1$.

Fix $x_{n+1} \in \mathbb{R}$ for now. By Hölder inequality and the induction hypothesis,

$$\begin{aligned}&\int_{\mathbb{R}^n} |f| \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \\ &\quad \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f_1 \ldots f_n|^{\frac{n}{n-1}} \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \right)^{\frac{n-1}{n}} \\ &\quad \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^{n} \left\| \tilde{f}_i \right\|_{L^n(\mathbb{R}^{n-1})} \qquad [\tilde{f}_i \text{ is } f_i \text{ with } x_{n+1} \text{ fixed}]\end{aligned}$$

Integrating w.r.t x_{n+1} and Hölder inequality gives the result. □

Proof. First we prove for $p = 1$.

We can assume $u \in C_c^{\infty}(\mathbb{R}^n)$. We have, for each $1 \leq i \leq n$,

$$|u(x_1, \ldots, x_n)| \leq \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \right| \, \mathrm{d}t := f_i(\hat{x}_i).$$

Thus, we have

$$|u(x_1, \ldots, x_n)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} |f_i(\hat{x}_i)|^{\frac{1}{n-1}}$$

Integrating and using the lemma, we deduce

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \, \mathrm{d}x \leq \prod_{i=1}^{n} \left\| |f_i(\hat{x}_i)|^{\frac{1}{n-1}} \right\|_{L^{n-1}(\mathbb{R}^{n-1})} \leq \prod_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}}$$

Thus,

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}} \leq c \sum_{i}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)} = c \, \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

This proves the case $p = 1$.

Now we choose $f = |u|^\gamma$ for some $\gamma > 0$ and apply the inequality for $p = 1$ to f to deduce,

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} \mathrm{d}x \leq \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u| \; \mathrm{d}x$$
$$\overset{\text{Hölder}}{\leq} \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}} \; \mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \; \mathrm{d}x\right)^{\frac{1}{p}}.$$

Now choose $\gamma > 0$ such that

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$$

and watch the exponents almost magically fall into place for $1 < p < n$ to establish

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \mathrm{d}x \leq c \left(\int_{\mathbb{R}^n} |\nabla u|^p \; \mathrm{d}x\right)^{\frac{1}{p}}.$$

This proves the theorem. □

Consequences of the Gagliardo-Nirenberg-Sobolev inequality

We now discuss some consequences of the inequality.

Theorem 3 (Sobolev embedding in $\mathbb{R}^n$ for $p < n$)**.** *Let $1 \leq p < n$. Then $W^{1,p}(\mathbb{R}^n)$ continuously embeds into $L^q(\mathbb{R}^n)$ for every $q \in [p, p^*]$.*

Proof. Since $q \in [p, p^*]$, we have

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*} \qquad \text{for some } \alpha \in [0,1].$$

Thus, we have, by interpolation inequality and Youngs inequality,

$$\|u\|_{L^q} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^{p^*}}^{1-\alpha} \leq \|u\|_{L^p} + \|u\|_{L^{p^*}} \leq c \|u\|_{W^{1,p}}.$$

□

Our next result might seem surprising, since it concerns $W^{1,n}$. But this result is more of a corollary of the proof and not really of the final result.

Theorem 4 (Sobolev embedding in $\mathbb{R}^n$ for $p = n$)**.** *$W^{1,n}(\mathbb{R}^n)$ continuously embeds into $L^q(\mathbb{R}^n)$ for every $q \in [n, +\infty)$.*

Proof. As in the proof, we can easily establish, for any $\gamma > 0$,

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} \mathrm{d}x \leq \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)n}{n-1}} \; \mathrm{d}x\right)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} |\nabla u|^n \; \mathrm{d}x\right)^{\frac{1}{n}}.$$

Note we really have not used the fact $p < n$ up to that point and so we can put $p = n$. Now let us chose $\gamma = n$. This will prove

$$\|u\|_{L^{\frac{n^2}{n-1}}} \leq c \|u\|_{W^{1,n}} .$$

But now we can iterate this process by choosing $\gamma = n+1, n+2, \ldots$ and so on to keep pushing the exponent. $\square$

Now we focus on bounded domains.

Theorem 5 (Sobolev embedding in bounded domains for $p < n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth and let $1 \leq p < n$. Then $W^{1,p}(\Omega)$ continuously embeds into $L^q(\Omega)$ for every $1 \leq q \leq p^*$.*

Theorem 6 (Sobolev embedding in $\mathbb{R}^n$ for $p = n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and smooth. Then $W^{1,n}(\Omega)$ continuously embeds into $L^q(\mathbb{R}^n)$ for every $1 \leq q < \infty$.*

Both results can be proved from the $\mathbb{R}^n$ case using extension and noting that Ω has finite measure.

Poincaré-Sobolev inequalities

Note that the Gagliardo-Nirenberg-Sobolev inequality actually says

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^p(\mathbb{R}^n)} \qquad \text{when } 1 \leq p < n.$$

However, the estimate in the result for the bounded, smooth domain says something weaker, namely,

$$\|u\|_{L^{p^*}(\Omega)} \leq c \|u\|_{W^{1,p}(\Omega)} \qquad \text{when } 1 \leq p < n.$$

It is in general not possible to improve this. But for functions in $W_0^{1,p}(\Omega)$, we can improve the inequality.

Theorem 7 (Poincaré-Sobolev inequality for $W_0^{1,p}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p < n$. Then there exists a constant $c > 0$, depending only on Ω, n and p such that we have the estimate*

$$\|u\|_{L^{p^*}(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} \qquad \textit{for all } u \in W_0^{1,p}(\Omega).$$

Remark 8. *Ω can be an arbitrary open set!*

The result follows from the Gagliardo-Nirenberg-Sobolev inequality by an extension, but not the extension operator we constructed in the theorem. There is a far simpler canonical extension operator for $W_0^{1,p}$ which is not available for $W^{1,p}$. This is the *extension by zero.* So the Poincaré-Sobolev inequality would follow easily as soon as we show the following simple lemma, whose proof is skipped.

Lemma 9. *Let $\Omega \subset \mathbb{R}^n$ be open and let $1 \leq p < \infty$. Then for any $u \in W_0^{1,p}(\Omega)$, the function*

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

is in $W^{1,p}(\mathbb{R}^n)$ and obviously extends u to whole of $\mathbb{R}^n$.

Remark 10. *Note that this lemma needs no regularity of the boundary and also does not need Ω to be bounded. However, if $\partial\Omega$ is not regular, there may be no well-defined trace and the identification with zero-trace functions might be meaningless.*

From the Poincaré-Sobolev inequality for $W_0^{1,p}$, we can now deduce

Theorem 11 (Poincaré inequality for $W_0^{1,p}$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $1 \leq p < \infty$. Then there exists a constant $c > 0$, depending only on Ω, n and p such that we have the estimate*

$$\|u\|_{L^p(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

Remark 12. *This shows that for any $\Omega \subset \mathbb{R}^n$ open and bounded, $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$. It is also fairly straight forward to establish that this implies that*

$$\langle u, v \rangle_{W_0^{1,2}(\Omega)} := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

is an equivalent inner product on $W_0^{1,2}(\Omega)$.