Introduction to the Calculus of Variations
Lecture Notes
Lecture 13

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Chapter 1

Sobolev spaces

1.1 Definitions

1.2 Elementary properties

1.3 Approximation and extension

1.3.1 Approximation and extension

In this section we are going to study two results.

- The **extension** of Sobolev functions from a bounded **smooth** domain to the whole of $\mathbb{R}^n$ while keeping control over the Sobolev norm.
- The **approximation** of a Sobolev function in a bounded **smooth** domain, in the Sobolev norm, by functions which are **smooth up to the boundary**.

Both the results can be proved for Lipschitz domains, but here we shall be content with smooth domains.

The approximation result actually follows from the extension result. Also we are going to explain the ideas involved for the proof of the extension result. The details can be and should be filled in easily. But first we state the results.

Extension

Theorem 1 (Extension operator)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Then for any $1 \leq p \leq \infty$, there exists a linear extension operator*

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that for all $u \in W^{1,p}(\Omega)$, *we have*

$$Pu|_{\Omega} = u, \tag{1.1}$$
$$\|Pu\|_{L^p(\mathbb{R}^n)} \leq c\,\|u\|_{L^p(\Omega)}, \tag{1.2}$$
$$\|Pu\|_{W^{1,p}(\mathbb{R}^n)} \leq c\,\|u\|_{W^{1,p}(\Omega)}, \tag{1.3}$$

where the constant $c > 0$ *depends only on* Ω.

Global approximation by smooth functions

Theorem 2 (Global approximation by smooth functions)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded with smooth boundary. Let* $u \in W^{1,p}(\Omega)$ *for some* $1 \leq p < \infty$. *Then there exists a sequence*

$$\{u_s\}_{s=1}^{\infty} \subset W^{1,p}(\Omega) \cap C^{\infty}\left(\overline{\Omega}\right)$$

such that

$$u_s \to u \qquad \text{in } W^{1,p}(\Omega).$$

Remark 3. *The result is false for* $p = \infty$.

Clearly, this result follows from the extension result by mollification.

Flattening the boundary

The first idea is that as the boundary $\partial\Omega$ is smooth, it is possible to locally **'flatten'** the boundary. In precise terms, if $x_0 \in \partial\Omega$, there exists a neighborhood $U_{x_0} \subset \mathbb{R}^n$ of x_0 such that there exists a **smooth diffeomorphism**

$$\Phi : \overline{B_1(0)} \to \overline{U_{x_0}}$$

satisfying

- $\Phi(0) = x_0$.
- $\Phi\left(B_1^+(0)\right) = \Omega \cap U_{x_0}$.
- $\Phi\left(B_1(0) \cap \{x_n = 0\}\right) = \partial\Omega \cap U_{x_0}$.

This basically is the coordinate change that maps the point x_0 to the origin, maps the portion of Ω in U_{x_0} to the upper half ball $B_1^+(0)$, maps the portion of $\partial\Omega$ in U_{x_0} to the portion of the equatorial hyperplane in the unit ball and takes the inward normal to $\partial\Omega$ to the postive direction of the x_n coordinate.

Covering the boundary by local patches

Thus, if we care only about a small neighborhood of a boundary point, we can transfer our problem to extending from the upper half-ball to the whole ball and then transfer back.

Now the question is, can we somehow 'cut' u into pieces near the boundary, work with each piece separately and then finally patch them up?

We can! But first we 'cut' the boundary into pieces.

Note that by compactness of $\partial\Omega$, it is possible to cover $\partial\Omega$ by **finitely many** such neighborhoods, i.e.

$$\partial\Omega \subset \bigcup_{i=1}^{M} U_{x_i},$$

for some integer $M > 0$ and some neighborhoods U_{x_i} of the boundary points $x_1, \ldots, x_M \in \partial\Omega$.

Localizing and patching them up

To 'cut' u into pieces, we use an extremely useful device known as a **partition of unity**.

Proposition 4 (partition of unity). *Let Γ be a compact subset of $\mathbb{R}^n$ and let $U_1, \ldots, U_M$ be a finite open covering of Γ. Then there exist functions $\zeta_0, \zeta_1, \ldots, \zeta_M \in C^\infty(\mathbb{R}^n)$ such that*

- $0 \le \zeta_i \le 1$ *for all* $0 \le i \le M$ *and*

$$\sum_{i=0}^{M} \zeta_i \equiv 1 \qquad \text{on } \mathbb{R}^n,$$

- $\operatorname{supp}\zeta_0 \subset \mathbb{R}^n \setminus \Gamma$ *and*
- $\operatorname{supp}\zeta_i$ *is compact and* $\operatorname{supp}\zeta_i \subset U_i$ *for every* $1 \le i \le M$.

Moreover, if $\Omega \subset \mathbb{R}^n$ is an open bounded set such that $\Gamma = \partial\Omega$, then we can in addition arrange that

$$\zeta_0\big|_\Omega \in C_c^\infty(\Omega).$$

Localizing or cutting into pieces How do we use this to 'cut' u into pieces?

Note that we have

$$u = \sum_{i=0}^{M} \zeta_i u \qquad \text{in } \Omega.$$

Thus,

$$u_i := \zeta_i u \qquad \text{for } 0 \le i \le M,$$

are the pieces of u.

Patching up the pieces On the other hand, if we are given functions $v_i \in W^{1,p}(U_i)$ for every $0 \leq i \leq M$, then

$$v := \sum_{i=0}^{M} \zeta_i v_i \in W^{1,p}(\mathbb{R}^n).$$

Sketch of the proof of the extension result

Now hopefully it is clear what we want to do. Our plan is

- cut u into pieces as discussed,
- locally flatten the boundary, i.e. compose each piece of u near the boundary with the respective diffemorphisms to obtain Sobolev functions defined on the upper half ball,
- extend those Sobolev functions to the whole ball,
- compose those extensions with the inverse of the diffeomorphisms to get Sobolev extensions to whole of the neighborhoods U_is and finally
- patch all these pieces together to obtain a Sobolev function on the whole of $\mathbb{R}^n$.

Note that the u_0 piece lives in the interior of Ω, so we can just extend it by zero.

To carry out the plan, all that remains is to figure out how to extend a Sobolev function on the upper half ball **which vanishes near the curved part of the boundary** from the upper half ball to the whole ball.

Extension by reflection

At the level of $W^{1,p}$, it is hardly surprising or difficult. We just use reflection across the flat part of the boundary of the upper half ball.

Lemma 5. *Let $u \in W^{1,p}\left(B_1^+(0)\right)$, where $1 \leq p \leq \infty$ be such that it vanishes near the curved part of the boundary, i.e. $\partial B_1(0) \cap \{x_n > 0\}$. Then the function defined as*

$$\tilde{u}(x', x_n) := \begin{cases} u(x', x_n) & \text{if } x_n > 0, \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

belongs to $W^{1,p}(B_1(0))$, extends u to $B_1(0)$ and vanishes near $\partial B_1(0)$.

The value of the function obviously matches and perhaps slightly less obviously, the tangential derivatives along the equatorial hyperplane match too. So the only thing to check is whether the normal derivative matches across the equatorial hyperplane $\{x_n = 0\}$. You are asked to prove this in the problem sheets.

1.4 Traces

Now we want to tackle the problem of defining ‘boundary values’ of a $W^{1,p}$ function. Note that $\partial\Omega$ has zero n-dimensional Lebesgue measure and so it is meaningless of talk about the ‘value’ of an L^p function on $\partial\Omega$. However, we shall see that unlike the case of a general L^p function, there is a precise sense in which we can define ‘values’ of a $W^{1,p}$ function on $\partial\Omega$.

Theorem 6 (Existence of Trace operator)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary and let $1 \leq p < \infty$. There exists a bounded linear operator*

$$T_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

$$T_0 u = u|_{\partial\Omega}$$

for any $u \in W^{1,p}(\Omega) \cap C^\infty\left(\overline{\Omega}\right)$.

For any $u \in W^{1,p}(\Omega)$, we call $T_0 u$ as the **zeroth order Dirichlet trace** on the boundary and is often denoted simply as $u|_{\partial\Omega}$.

Remark 7.

- *Note that $L^p(\partial\Omega)$ is defined with respect to the surface measure $d\sigma$ on $\partial\Omega$. If you are familiar with Hausdorff measure, then you would have no difficulty understanding that this is essentially the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ restricted to $\partial\Omega$.*
- *Although we prove the theorem for $1 \leq p < \infty$, this* ***does not*** *mean that $W^{1,\infty}$ functions have no bounded trace. In fact, as we shall see later, $W^{1,\infty}(\Omega) = C^{0,1}\left(\overline{\Omega}\right)$. Thus, being Lipschitz, these functions have boundary values in the usual sense and those are clearly bounded on the compact set $\partial\Omega$.*
- *In fact, $T_0 u = u|_{\partial\Omega}$ for any $u \in W^{1,p}(\Omega) \cap C^0\left(\overline{\Omega}\right)$.*
- *As before, smooth boundary is not really necessary and the result holds for Lipschitz boundaries as well. But some regularity of the boundary is essential.*

We define the operator for smooth functions by assigning the boundary values. This operator is clearly linear. Since $C^\infty\left(\overline{\Omega}\right)$ is dense in $W^{1,p}(\Omega)$, to show boundedness, it is enough to show the estimate

$$\left\| u|_{\partial\Omega} \right\|_{L^p(\partial\Omega)} \leq c \left\| u \right\|_{W^{1,p}(\Omega)} \qquad \text{for every } u \in C^\infty\left(\overline{\Omega}\right).$$

As before, by using localization, flattening the boundary and patching up, the proof of the theorem can be reduced to proving the following.

Lemma 8. *There exists a constant $c > 0$ such that*

$$\left(\int_{\mathbb{R}^{n-1}} \left| u(x',0) \right|^p \, \mathrm{d}x' \right)^{\frac{1}{p}} \leq c \left\| u \right\|_{W^{1,p}\left(\mathbb{R}^n_+\right)} \qquad \textit{for every } u \in C_c^\infty(\mathbb{R}^n).$$

Proof.

Let $$F(t) := |t|^{p-1} t, \quad \text{for all } t \in \mathbb{R}.$$

Now since $u \in C_c^{\infty}(\mathbb{R}^n)$, we have

$$F(u(x',0)) = -\int_0^{+\infty} \frac{\partial}{\partial x_n} F(u(x',x_n))\, \mathrm{d}x_n \quad = -\int_0^{+\infty} F'(u(x',x_n)) \frac{\partial u}{\partial x_n}(x',x_n)\, \mathrm{d}x_n$$

Taking absolute values and then Young's inequality, this implies

$$\begin{aligned} |u(x',0)|^p &\leq p \int_0^{+\infty} |u(x',x_n)|^{p-1} \left| \frac{\partial u}{\partial x_n}(x',x_n) \right| \mathrm{d}x_n \\ &\leq c \left(\int_0^{+\infty} |u(x',x_n)|^p \, \mathrm{d}x_n + \int_0^{+\infty} \left| \frac{\partial u}{\partial x_n}(x',x_n) \right|^p \mathrm{d}x_n \right). \end{aligned}$$

The lemma follows by integrating w.r.t. $x' \in \mathbb{R}^{n-1}$ and taking p-th roots along with obvious estimates. □

- One can now easily figure out the kernel of the trace map.
$$\mathrm{Ker}(T_0) = W_0^{1,p}(\Omega).$$
- Figuring out the exact image of the trace map is delicate. They are however known, but requires the notion of Sobolev spaces of fractional order. For example,
$$T_0\left(W^{1,p}(\Omega)\right) = W^{1-\frac{1}{p},p}(\partial\Omega).$$
- Higher order traces can be defined similarly and requires more Sobolev regularity for those traces to be in $L^p(\partial\Omega)$. For example, for $u \in W^{2,p}(\Omega)$, there is a bounded linear operator
$$T_1 : W^{2,p}(\Omega) \to L^p(\partial\Omega)$$
such that
$$T_1 u = \left. \frac{\partial u}{\partial x_n} \right|_{\partial\Omega}$$
for any $u \in W^{2,p}(\Omega) \cap C^{\infty}\left(\overline{\Omega}\right)$.

1.5 Sobolev inequalities and Sobolev embeddings

1.5.1 Sobolev embeddings

Now we are going to study an extremely important topic in the theory of Sobolev spaces, called the **Sobolev embeddings**.

A $W^{1,p}$ function is apriori only in L^p. Now we ask the question if the additional information that the weak derivative is also in L^p implies that the function enjoys better integrability, i.e. actually is in L^q for some $q > p$?

There are three different regimes in this discussion, depending on what n and p is.

- **Sobolev inequality** The case $1 \leq p < n$.
- **Morrey's inequality** The case $n < p < \infty$.
- The borderline case $p = n$.

We begin our discussion with the case $1 \leq p < n$.

1.5.2 Gagliardo-Nirenberg-Sobolev inequalities

Sobolev conjugate exponent for $1 \leq p < n$

Suppose we want to prove an inequality of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n). \tag{1.4}$$

What can the exponent q be? To guess this, we perform what is known as a **scaling analysis**. If such a inequality is indeed true, then choose $u \in C_c^\infty(\mathbb{R}^n)$, $u \not\equiv 0$ and $\lambda > 0$ and set

$$u_\lambda(x) := u(\lambda x).$$

Thus, we deduce, using change of variables

$$\int_{\mathbb{R}^n} |u_\lambda|^q \, \mathrm{d}x = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u|^q \, \mathrm{d}x$$

$$\int_{\mathbb{R}^n} |\nabla u_\lambda|^p \, \mathrm{d}x = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x.$$

So, (1.4) applied to u_λ implies

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c\lambda^{\left(1-\frac{n}{p}+\frac{n}{q}\right)} \|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{1.5}$$

Sobolev conjugate exponent for $1 \leq p < n$

Thus, if

$$1 - \frac{n}{p} + \frac{n}{q} \neq 0,$$

we can easily contradict the inequality by letting λ go to 0 or ∞. Thus, we must have

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

Definition 9 (Sobolev conjugate exponent)**.** *Let $1 \leq p < n$. Then the **Sobolev conjugate exponent** of p is defined as*

$$p^* = \frac{np}{n-p}.$$

Remark 10. *Note that we always have $p^* > p$.*

Gagliardo-Nirenberg-Sobolev inequality

Theorem 11 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let* $1 \leq p < n$*. Then there exists a constant* $c > 0$*, depending only on* n *and* p *such that we have the estimate*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\right)^{\frac{1}{p^*}} \leq c \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}} \tag{1.6}$$

for every $u \in W^{1,p}(\mathbb{R}^n)$.