Introduction to the Calculus of Variations
Lecture Notes
Lecture 12

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Chapter 1

Prelude to Direct Methods

1.1 Geodesics: the problem

1.2 Absolute continuity: first encounter with Sobolev spaces

1.3 Existence of geodesics

1.4 Regularity questions

1.4.1 Regularity of minimizers

1.4.2 Geodesic equation

As a consequence of the regularity theorem, we infer that the geodesic curves are smooth. Hence they satisfy the Euler-Lagrange equations, which we now deduce. The functional is

$$\boxed{E\left(c\right)=\frac{1}{2}\int_{0}^{T}g_{ij}\left(\gamma\left(t\right)\right)\dot{\gamma^{i}}\left(t\right)\dot{\gamma^{j}}\left(t\right)\ \mathrm{d}t.}$$

Thus, we calculate

$$\begin{aligned}0&=\frac{d}{dt}E_{\dot{\gamma}_{i}}-E_{\gamma_{i}}\\&=\frac{d}{dt}\left[2g_{ij}\left(\gamma\left(t\right)\right)\dot{\gamma^{j}}\left(t\right)\right]-\left(\frac{\partial}{\partial z^{i}}g_{kj}\right)\left(\gamma\left(t\right)\right)\dot{\gamma^{k}}\left(t\right)\dot{\gamma^{j}}\left(t\right)\\&=2g_{ij}\ddot{\gamma^{j}}+2\frac{\partial}{\partial z^{k}}g_{ij}\dot{\gamma^{k}}\dot{\gamma^{j}}-\frac{\partial}{\partial z^{i}}g_{kj}\dot{\gamma^{k}}\dot{\gamma^{j}}.\end{aligned}$$

Writing g^{ij} as the entries of the inverse matrix $(g_{ij})_{i,j}$ and using the notation

$$g_{ij,k} := \frac{\partial}{\partial z^k} g_{ij},$$

the EL equations become

$$\boxed{\ddot{\gamma}^i(t) + \Gamma^i_{jk}(\gamma(t))\,\dot{\gamma}^j(t)\,\dot{\gamma}^k(t) = 0,}$$

where

$$\boxed{\Gamma^i_{jk} := \frac{1}{2} g^{il}\left(g_{jl,k} + g_{kl,j} - g_{jk,l}\right),}$$

are called the **Christoffel symbols**. Often in differential geometry courses, existence of geodesic is proved via classical method, i.e. by applying the existence of solutions of ODE theorem to this ODE.

Chapter 2

Sobolev spaces

2.1 Definitions

Now we are going to enter the arena of modern direct methods. But before doing so, we must get ourselves acquainted with Sobolev spaces. We have already seen Sobolev spaces in dimension one. Now we are going study Sobolev spaces in general dimensions.

We start with notion of weak derivatives.

Definition 1. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{loc}(\Omega)$. We say that $v \in L^1_{loc}(\Omega)$ is the* weak partial derivative *of u with respect to x_i if*

$$\int_\Omega v(x)\,\varphi(x)\,dx = -\int_\Omega u(x)\,\frac{\partial \varphi}{\partial x_i}(x)\,dx, \quad \forall\, \varphi \in C_c^\infty(\Omega).$$

By abuse of notations we write $v = \partial u/\partial x_i$ or u_{x_i}.

We say that u is weakly differentiable *if the weak partial derivatives $u_{x_1}, \cdots,$ u_{x_n} exist.*

Remark 2. ***(i)*** *If such a weak derivative exists it is unique (a.e.), as a consequence of the fundamental lemma of calculus of variations.*

(ii) *All the usual rules of differentiation are easily generalized to the present context of weak differentiability.*

(iii) *In a similar way we can introduce the higher derivatives.*

(iv) *If a function is C^1, then the usual notion of derivative and the weak one coincide.*

(v) *The advantage of this notion of weak differentiability will be obvious when defining Sobolev spaces. We can compute many more derivatives of functions than one can usually do. However, not all measurable functions can be differentiated in this way. In particular, a discontinuous function of $\mathbb{R}$ cannot be differentiated in the weak sense (as we have already seen, $W^{1,1}$ functions on $\mathbb{R}$ are continuous.)*

Example 3. *Let* $\Omega = \mathbb{R}$ *and the function* $u(x) = |x|$. *Its weak derivative is then given by*

$$u'(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Example 4 (Dirac mass). *Let*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We now show that H *has no weak derivative. Let* $\Omega = (-1, 1)$. *Assume, for the sake of contradiction, that* $H' = \delta \in L^1_{loc}(-1, 1)$ *and let us prove that this is impossible. Let* $\varphi \in C_c^\infty(0, 1)$ *be arbitrary and extend it to* $(-1, 0)$ *by* $\varphi \equiv 0$. *We therefore have by definition that*

$$\int_{-1}^{1} \delta(x)\varphi(x)\,dx = -\int_{-1}^{1} H(x)\varphi'(x)\,dx = -\int_{0}^{1} \varphi'(x)\,dx$$
$$= \varphi(0) - \varphi(1) = 0.$$

We hence find

$$\int_0^1 \delta(x)\varphi(x)\,dx = 0, \quad \forall\, \varphi \in C_c^\infty(0, 1)$$

which implies $\delta = 0$ a.e. in $(0, 1)$. With an analogous reasoning we would get that $\delta = 0$ a.e. in $(-1, 0)$ and consequently $\delta = 0$ a.e. in $(-1, 1)$. Let us show that we have reached the desired contradiction. Indeed, if this were the case we would have, for every $\varphi \in C_c^\infty(-1, 1)$,

$$0 = \int_{-1}^{1} \delta(x)\varphi(x)\,dx = -\int_{-1}^{1} H(x)\varphi'(x)\,dx$$
$$= -\int_0^1 \varphi'(x)\,dx = \varphi(0) - \varphi(1) = \varphi(0).$$

This would imply that $\varphi(0) = 0$, for every $\varphi \in C_c^\infty(-1, 1)$, which is clearly absurd. Thus H is not weakly differentiable. By weakening even more the notion of derivative (for example, by no longer requiring that v is in L^1_{loc}), the theory of distributions can give a meaning at $H' = \delta$, it is then called the *Dirac mass*. We will however not need this theory.

Definition 5 (Sobolev spaces). *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $1 \leq p \leq \infty$.

(i) *We let* $W^{1,p}(\Omega)$ *be the set of functions* $u : \Omega \to \mathbb{R}$, $u \in L^p(\Omega)$, *whose weak partial derivatives* $u_{x_i} \in L^p(\Omega)$ *for every* $i = 1, \cdots, n$. *We endow this space with the following norm*

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p\right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$

$$\|u\|_{W^{1,\infty}} = \max\{\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\} \quad \text{if } p = \infty.$$

(ii) *If $1 \le p < \infty$, the set $W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ functions in $W^{1,p}(\Omega)$. By abuse of language, we often say, if Ω is bounded, that $u \in W_0^{1,p}(\Omega)$ is such that $u \in W^{1,p}(\Omega)$ and $u = 0$ on $\partial\Omega$.*

(iii) *We also write $u \in u_0 + W_0^{1,p}(\Omega)$, meaning that $u, u_0 \in W^{1,p}(\Omega)$ and $u - u_0 \in W_0^{1,p}(\Omega)$.*

(iv) *We let $W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$.*

(v) Analogously we define the Sobolev spaces with higher derivatives as follows. If $k > 0$ is an integer we let $W^{k,p}(\Omega)$ be the set of functions $u : \Omega \to \mathbb{R}$, whose weak partial derivatives $D^a u \in L^p(\Omega)$, for every multi-index $a \in \mathcal{A}_m$, $0 \le m \le k$. The norm is given by

$$\|u\|_{W^{k,p}} = \begin{cases} \left(\sum_{0\le|a|\le k} \|D^a u\|_{L^p}^p\right)^{1/p} & \text{if } 1 \le p < \infty \\ \max_{0\le|a|\le k} \left(\|D^a u\|_{L^\infty}\right) & \text{if } p = \infty. \end{cases}$$

(vi) If $1 \le p < \infty$, $W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and $W_0^{k,\infty}(\Omega) = W^{k,\infty}(\Omega) \cap W_0^{k,1}(\Omega)$.

(vii) We define $W^{k,p}(\Omega; \mathbb{R}^N)$ to be the set of maps $u : \Omega \to \mathbb{R}^N$, $u = (u^1, \cdots, u^N)$, with $u^i \in W^{k,p}(\Omega)$ for every $i = 1, \cdots, N$ and similarly for $W_0^{k,p}(\Omega; \mathbb{R}^N)$.

Remark 6. ***(i)*** *By abuse of notations we write $W^{0,p} = L^p$.*

(ii) *Roughly speaking, we can say that $W^{1,p}$ is an extension of C^1 similar to that of L^p as compared to C^0.*

(iii) *Note that if Ω is bounded, then*

$$C^1(\overline{\Omega}) \underset{\neq}{\subset} W^{1,\infty}(\Omega) \underset{\neq}{\subset} W^{1,p}(\Omega) \underset{\neq}{\subset} L^p(\Omega)$$

for every $1 \le p < \infty$.

(iv) *If $p = 2$, the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are sometimes respectively denoted by $H^k(\Omega)$ and $H_0^k(\Omega)$.*

Examples

Now we present some examples which are instructive.

Example 7. *Let $s > 0$,*

$$\Omega = \{x \in \mathbb{R}^n : |x| < 1\} \quad \text{and} \quad \psi(x) = |x|^{-s}.$$

We then have

$$\psi \in L^p \Leftrightarrow sp < n \quad \text{and} \quad \psi \in W^{1,p} \Leftrightarrow (s+1)p < n.$$

This is a typical one about how singular a Sobolev function can be. Another similar one is the following.

Example 8. *Let $n \geq 2$ and*

$$\Omega = \{x \in \mathbb{R}^n : |x| < 1\} \quad \text{and} \quad u(x) = \frac{x}{|x|},$$

then $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ for every $1 \leq p < n$.

We already saw $W^{1,1}$ functions for $n = 1$ are continuous. Our next example shows $W^{1,n}$ functions need not be continuous if $n \geq 2$.

Example 9. *Let $0 < s < 1/2$,*

$$\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1/2\right\} \quad \text{and} \quad \psi(x) = |\log |x||^s.$$

We have that $\psi \in W^{1,2}(\Omega)$, $\psi \in L^p(\Omega)$ for every $1 \leq p < \infty$, but $\psi \notin L^\infty(\Omega)$.

Another such example is

Example 10. $u(x) = \log\log\left(1 + \frac{1}{|x|}\right) \in W^{1,n}(B_1^n)$ *for $n > 1$, but is unbounded near* 0.

2.2 Elementary properties

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p \leq \infty$ and $k \geq 1$ an integer.*

(i) *$W^{k,p}(\Omega)$ equipped with its norm $\|\cdot\|_{k,p}$ is a Banach space which is **separable** if $1 \leq p < \infty$ and **reflexive** if $1 < p < \infty$.*

(ii) *$W^{1,2}(\Omega)$ is a Hilbert space when endowed with the following inner product*

$$\langle u; v\rangle_{W^{1,2}} = \int_\Omega u(x)\, v(x)\, dx + \int_\Omega \langle \nabla u(x); \nabla v(x)\rangle\, dx.$$

(iii) *The $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ functions are dense in $W^{k,p}(\Omega)$ provided $1 \leq p < \infty$. Moreover, if Ω is a bounded connected open set with Lipschitz boundary, then $C^\infty\left(\overline{\Omega}\right)$ is also dense in $W^{k,p}(\Omega)$ provided $1 \leq p < \infty$.*

(iv) *$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$, whenever $1 \leq p < \infty$.*

Remark 12. ***(i)*** *Note that as for the case of L^p the space $W^{k,p}$ is reflexive only when $1 < p < \infty$ and hence $W^{1,1}$ is not reflexive.*

(ii) *The density result for arbitrary open sets is due to Meyers and Serrin, known as Meyer-Serrin theorem and is quite delicate. The result for domains with regular enough boundary is easier and is proved by extension and mollification, as in the one dimensional case. But this case is harder than the one dimensional case.*

(iii) *In general, we have $W_0^{1,p}(\Omega) \underset{\neq}{\subset} W^{1,p}(\Omega)$, but when $\Omega = \mathbb{R}^n$ both coincide.*

We are not going to prove the Meyer-Serrin theorem. The proof of the density result with regular enough boundary would be sketched later. Now we first prove a simple characterization of $W^{1,p}$ which turns out to be particularly helpful when dealing with regularity problems. It relates the weak derivative with the difference quotient that characterizes classical derivatives. First we begin with a notation for difference quotients.

Notation: For $\tau \in \mathbb{R}^n$, $\tau \neq 0$, we let the *difference quotient* be defined by

$$(D_\tau u)(x) = \frac{u(x+\tau) - u(x)}{|\tau|} .$$

Note that if u is C^1, the limits of difference quotients are the classical derivatives. So we can expect that the difference quotients are also in L^p for $W^{1,p}$ functions, since the weak derivative is in L^p. This is, in fact, true and it actually characterizes $W^{1,p}$.

Theorem 13 (Characterization of difference quotients)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p \leq \infty$ and $u \in L^p(\Omega)$. The following properties are then equivalent.*

(i) *$u \in W^{1,p}(\Omega)$;*

(ii) *there exists a constant $\gamma = \gamma(u, \Omega, p)$ so that*

$$\left| \int_\Omega u\, \varphi_{x_i} \right| \leq \gamma \|\varphi\|_{L^{p'}} , \quad \forall\, \varphi \in C_c^\infty(\Omega) ,\ \forall\, i = 1, 2, \cdots, n$$

(iii) *there exists a constant $\gamma = \gamma(u, \Omega, p)$ such that for every open set $\omega \subset \overline{\omega} \subset \Omega$, with $\overline{\omega}$ compact, and for every $\tau \in \mathbb{R}^n$ with $0 \neq |\tau| < \operatorname{dist}(\omega, \Omega^c)$ (where $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$), we have*

$$\|D_\tau u\|_{L^p(\omega)} \leq \gamma.$$

Furthermore, if (ii) or (iii) holds, then

$$\|\nabla u\|_{L^p(\Omega)} \leq \gamma.$$

If (i) holds, then γ in (ii) or (iii) can be chosen to be $\|\nabla u\|_{L^p}$ and in particular

$$\|D_\tau u\|_{L^p(\omega)} \leq \|\nabla u\|_{L^p(\Omega)} .$$

Remark 14. ***(i)*** *As a consequence of the theorem, it can easily be proved that if Ω is bounded and open then*

$$C^{0,1}\left(\overline{\Omega}\right) \subset W^{1,\infty}(\Omega)$$

and the inclusion is, in general, strict. If, however, the set Ω is also convex (or sufficiently regular), then these two sets coincide (as usual, up to the choice of a representative in $W^{1,\infty}(\Omega)$). In other words, we can say that the set of Lipschitz *functions over $\overline{\Omega}$ can be identified, if Ω is convex or sufficiently regular, with the space $W^{1,\infty}(\Omega)$.*

(ii) *The theorem is false when $p = 1$. We then only have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). The functions satisfying (ii) or (iii) are then called functions of bounded variations, as we have already mentioned in the setting of Sobolev spaces in dimension one.*

Proof. We prove the theorem only when $n = 1$ and $\Omega = (a, b)$. Adapting the proofs to the more general case is straight forward and is given as an exercise in the problem sheet.

(i) $\Rightarrow$ (ii). This follows from Hölder inequality and the fact that u has a weak derivative; indeed

$$\left|\int_a^b u\,\varphi'\right| = \left|\int_a^b u'\,\varphi\right| \leq \|u'\|_{L^p}\,\|\varphi\|_{L^{p'}}\ .$$

(ii) $\Rightarrow$ (i). Let F be a linear functional defined by

$$F(\varphi) = \langle F;\varphi\rangle = \int_a^b u\,\varphi', \quad \forall\,\varphi \in C_c^\infty(a,b)\,. \tag{2.1}$$

Note that, by (ii), it is continuous over $C_c^\infty(a,b)$. Since $C_c^\infty(a,b)$ is dense in $L^{p'}(a,b)$ (note that we use here the fact that $p \neq 1$ and hence $p' \neq \infty$), we can extend it, by continuity (or appealing to Hahn-Banach theorem), to the whole $L^{p'}(a,b)$; we have therefore defined a continuous linear operator F over $L^{p'}(a,b)$, with

$$|F(\varphi)| \leq \gamma\,\|\varphi\|_{L^{p'}}\ , \quad \forall\,\varphi \in L^{p'}(a,b)\,. \tag{2.2}$$

Sobolev spaces: Properties From Riesz theorem representation theorem for L^p, we find that there exists $v \in L^p(a,b)$ so that

$$F(\varphi) = \langle F;\varphi\rangle = \int_a^b v\,\varphi, \quad \forall\,\varphi \in L^{p'}(a,b)\,. \tag{2.3}$$

Combining (2.1) and (2.3) we get

$$\int_a^b (-v)\,\varphi = -\int_a^b u\,\varphi', \quad \forall\,\varphi \in C_c^\infty(a,b)$$

which exactly means that $u' = -v \in L^p(a,b)$ and hence $u \in W^{1,p}(a,b)$.

Note also that, since (2.2) and (2.3) hold, we infer

$$\|u'\|_{L^p(a,b)} = \|v\|_{L^p(a,b)} \leq \gamma.$$

(iii) $\Rightarrow$ (ii). Let $\varphi \in C_c^\infty(a,b)$ and let $\omega \subset \overline{\omega} \subset (a,b)$ with $\overline{\omega}$ compact and such that $\operatorname{supp}\varphi \subset \omega$. Let $\tau \in \mathbb{R}$ so that $0 \neq |\tau| < \operatorname{dist}\left(\omega, (a,b)^c\right)$.

We then have for $1 < p \leq \infty$, appealing to Hölder inequality and to (iii),

$$\left| \int_a^b (D_\tau u)\, \varphi \right| \leq \|D_\tau u\|_{L^p(\omega)} \|\varphi\|_{L^{p'}(a,b)} \leq \gamma \|\varphi\|_{L^{p'}(a,b)} . \tag{2.4}$$

We know, by hypothesis, that $\varphi \equiv 0$ on $(a, a+\tau)$ and $(b-\tau, b)$ if $\tau > 0$ and we therefore find (letting $\varphi \equiv 0$ outside (a,b))

$$\int_a^b u(x+\tau)\, \varphi(x)\, dx = \int_{a+\tau}^{b+\tau} u(x+\tau)\, \varphi(x)\, dx = \int_a^b u(x)\, \varphi(x-\tau)\, dx. \tag{2.5}$$

Since a similar argument holds for $\tau < 0$, we deduce from (2.4) and (2.5) that, if $1 < p \leq \infty$,

$$\left| \int_a^b u(x)\, [\varphi(x-\tau) - \varphi(x)]\, dx \right| \leq \gamma\, |\tau|\, \|\varphi\|_{L^{p'}(a,b)} .$$

Letting $|\tau|$ tend to zero, we get

$$\left| \int_a^b u\, \varphi' \right| \leq \gamma \|\varphi\|_{L^{p'}(a,b)} , \quad \forall\, \varphi \in C_c^\infty(a,b)$$

which is exactly (ii).

Note that the γ appearing in (iii) and in (ii) can be taken the same.

(i) $\Rightarrow$ (iii). We have for every $x \in \omega$

$$u(x+\tau) - u(x) = \int_x^{x+\tau} u'(t)\, dt = \tau \int_0^1 u'(x+s\tau)\, ds$$

and hence

$$|u(x+\tau) - u(x)| \leq |\tau| \int_0^1 |u'(x+s\tau)|\, ds.$$

Let $1 < p < \infty$ (the conclusion is obvious if $p = \infty$), we have from Jensen inequality that

$$|u(x+\tau) - u(x)|^p \leq |\tau|^p \int_0^1 |u'(x+s\tau)|^p\, ds$$

Hence after integration

$$\begin{aligned} \int_\omega |u(x+\tau) - u(x)|^p\, dx &\leq |\tau|^p \int_\omega \int_0^1 |u'(x+s\tau)|^p\, ds\, dx \\ &= |\tau|^p \int_0^1 \int_\omega |u'(x+s\tau)|^p\, dx\, ds. \end{aligned}$$

Since $\omega + s\tau \subset (a, b)$, we find

$$\int_{\omega} |u'(x + s\tau)|^p \, dx = \int_{\omega + s\tau} |u'(y)|^p \, dy \leq \|u'\|_{L^p(a,b)}^p$$

and hence

$$\|D_\tau u\|_{L^p(\omega)} \leq \|u'\|_{L^p(a,b)}$$

which is the claim.

□