Introduction to the Calculus of Variations
Lecture Notes
Lecture 11

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Chapter 1

Prelude to Direct Methods

1.1 Geodesics: the problem

1.2 Absolute continuity: first encounter with Sobolev spaces

1.2.1 Poincaré inequality in $W_0^{1,p}$

Now we can show an important inequality known as the **Poincaré inequality**.

Theorem 1 (Poincaré inequality). *Let (a,b) be a bounded interval and let $u \in W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$ for $1 \leq p < \infty$. Then we have*

$$\int_a^b |u(t)|^p \, \mathrm{d}t \leq (b-a)^p \int_a^b |\dot{u}\,(t)|^p \, \mathrm{d}t.$$

In particular, the expression

$$\left(\int_a^b |\dot{u}\,(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}$$

is an equivalent norm (i.e equivalent to the $W^{1,p}$ norm) on $W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$.

Proof. We leave out the details of the proof and only provide a sketch, as this is fairly easy. We prove it for C_c^∞ functions first using the fundamental theorem of calculus and easy estimates and Hölder inequality. Then we claim the result for $W_0^{1,p}$ by density. □

1.2.2 Absolutely continuous functions

The geodesic problem was first solved for **absolutely continuous** curves, which were introduced by Vitalli. First we begin with a precise definition.

Definition 2 (absolutely continuous functions)**.** *A function $u:(a,b)\to\mathbb{R}$ is said to be absolutely continuous, denoted $u\in AC((a,b))$, if, for every $\varepsilon>0$, there is a $\delta>0$ such that*

$$\sum_{i=1}^{M}|\beta_i-\alpha_i|<\delta \quad \textit{implies} \quad \sum_{i=1}^{M}|u(\beta_i)-u(\alpha_i)|<\varepsilon$$

whenever $(\alpha_1,\beta_1),\ldots,(\alpha_M,\beta_M)$ are disjoint segments in (a,b).

Remark 3.

- *The vector-valued version is defined similarly.*
- *Clearly, any absolutely continuous function is uniformly continuous.*
- *Any absolutely continuous function is also of* ***bounded variation****. More precisely, if $u\in AC((a,b))$, we have*

  $$V_a^b(u):=\sup\sum_{i=1}^{M}|u(x_i)-u(x_{i-1})|<+\infty,$$

  where the supremum is taken over all natural numbers M and all choices of x_is such that $a<x_0<x_1<\ldots<x_M<b$.
- *However, much more is true. In fact, we have,*

  $$AC\left((a,b);\mathbb{R}^N\right)=W^{1,1}\left((a,b);\mathbb{R}^N\right).$$

  We shall prove it in the problem sheet in stages.

1.3 Existence of geodesics

Now we return to the problem of showing the existence of a geodesic. The variational problem is

$$\inf_{\gamma\in X}\left\{L(c):=\int_0^T\left(g_{ij}(\gamma(t))\,\dot{\gamma}^i(t)\,\dot{\gamma}^j(t)\right)^{\frac{1}{2}}\,\mathrm{d}t\right\}=m.$$

where

$$X=\left\{:\gamma\in C^1([0,T];U):\gamma(0)=f^{-1}(p_1),\gamma(T)=f^{-1}(p_2)\right\}.$$

Here we are implicitly making the identification of c with γ via a fixed local chart f.

We have already seen that this problem does not have enough compactness properties. We are going to inspect why in a bit more detail from another perspective.

1.3.1 Compactness and reparametrization

Invariance under reparametrization

The length functional is invariant under reparametrization. To see this, let

$$\tau : [0, S] \to [0, T]$$

be a **diffeomorphism**. Then we easily compute that

$$L(c) = L(c \circ \tau) \qquad \text{for any curve } c : [0, T] \to \mathbb{R}^N.$$

Indeed,

$$\begin{aligned} L(c \circ \tau) &= \int_0^S \left| \frac{d}{ds} \left(c \circ \tau\right)(s) \right| \, \mathrm{d}s \\ &= \int_0^S \left| \left(\frac{d}{dt} c \right) (\tau(s)) \right| \left| \frac{d\tau}{ds}(s) \right| \, \mathrm{d}s \\ &= \int_0^T \left| \dot{c}\,(t) \right| \, \mathrm{d}t = L\,(c)\,. \end{aligned}$$

Reparametrization and group action

What is happening here is a noncompactness due to a group action, here the group being the diffeomorphism group of an interval.

Just for an analogy, suppose we are looking to find the unit interval $[0, 1]$. Suppose, however, that our problem is invariant under the **action of** $\mathbb{Z}$, i.e. invariant under the transformation

$$x \mapsto x + \mathbb{Z}.$$

Now, the trouble is, though our problem does not distinguish between copies of the same interval, we do and thus instead of finding the compact interval $[0, 1]$, we would find the collection of all integer translated copies of the interval, which is $\mathbb{R}$(!) and is **noncompact**!

Probably it is better to view the analogy in reverse.

Suppose we are working with the **noncompact** space $\mathbb{R}$, and the lack of compactness causes trouble for us.

But suppose that our problem is invariant under the **action of** $\mathbb{Z}$.

Since the problem does not distinguish between copies of the same interval, we can cease to do so as well! Thus, we can 'quotient out' the invariance or pass to the quotient' $\mathbb{R}/\mathbb{Z}$.

Question: What is the quotient?

$$\mathbb{R}/\mathbb{Z} = \mathbb{S}^1,$$

which is **compact**! Caution: Incidentally, $\mathbb{S}^1$ is also the one-point compactification of $\mathbb{R}$. However, this is not at play here and is just incidental here. For example, if the domain is $\mathbb{R}^n$ for $n \geq 2$ and the problem is invariant under the transformation (translation by an **integer lattice**)

$$x \mapsto x + \mathbb{Z}^n,$$

then

$$\mathbb{R}^n / \mathbb{Z}^n = \mathbb{T}^n,$$

the n**-torus**, which is once again compact, but is quite different from the one point compactification of $\mathbb{R}^n$, which is $\mathbb{S}^n$.

Fixing the parametrization

To 'quotient out' this invariance, we need to fix a parametrization. But which one? To answer this, let us go back to the easier problem for which we have more hopes of solving. We define **the energy** of a curve c as

$$E(c) := \frac{1}{2} \int_0^T |\dot{c}(t)|^2 \, \mathrm{d}t = \frac{1}{2} \int_0^T g_{ij}(\gamma(t)) \dot{\gamma^i}(t) \dot{\gamma^j}(t) \, \mathrm{d}t$$

Now we notice that for $c \in W^{1,2}\left([0,T]; \mathbb{R}^N\right)$, we have,

$$\begin{aligned} L(c) &= \int_0^T |\dot{c}(t)| \, \mathrm{d}t \\ &\overset{\text{Hölder}}{\leq} \sqrt{T} \left(\int_0^T |\dot{c}(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \\ &\leq \sqrt{2T} \sqrt{E(c)}, \end{aligned}$$

with equality if and only if

$$|\dot{c}(t)| = \text{constant} \qquad \text{a.e.}$$

Parametrization by arc-length

Definition 4. *We say a curve $c \in AC\left([0,T]; \mathbb{R}^N\right)$ is* ***parametrized proportionally to arc-length*** *if it satisfies*

$$|\dot{c}(t)| = \textit{constant} \qquad \textit{a.e.}$$

We say the curve is ***parametrized by arc-length*** *if*

$$|\dot{c}(t)| = 1 \qquad \textit{a.e.}$$

Remark 5. • *Any Lipschitz curve can be (re)parametrized by arc-length.*

- *Any injective, rectifiable, absolutely continuous curve can be (re)parametrized by arc-length.*

Proposition 6. *Let $c : [0, L(c)] \to \mathbb{R}^N$ be a curve which can be parametrized by arc-lenth. Then among all reparametrizations*

$$\tau : [0, L(c)] \to [0, L(c)],$$

the parametrization by arc-length has the ***smallest*** *energy and satisfies*

$$L(c) = 2E(c).$$

Thus, we can minimize $E(c)$ instead of $L(c)$ to find geodesic curves.

1.3.2 Existence of geodesics

Now we settle the problem of the existence of a geodesic.

Theorem 7 (existence of geodesics)**.** *Assume $f : U \subset \mathbb{R}^N \to M \subset \mathbb{R}^d$ be a chart on M such that the metric tensor is uniformly positive definite in $f(U)$ and let $p_1 \neq p_2 \in M$ be contained in the image of f. Suppose there exists at least one Lipschitz curve c_0 on $f(U)$ joining p_1 and p_2 and let*

$$D := \inf\{L(c) : c \textit{ is a Lipschitz curve on } f(U) \textit{ joining } p_1 \textit{ and } p_2\}.$$

Then the variational problem

$$\inf_{\gamma \in \gamma_0 + W_0^{1,2}([0,D];U)} \left\{ I(\gamma) := \frac{1}{2}\int_0^D g_{ij}(\gamma(t))\,\dot{\gamma^i}(t)\,\dot{\gamma^j}(t)\;\mathrm{d}t \right\} = \frac{1}{2}D,$$

where $f \circ \gamma_0 = c_0$, has a minimizer.

Proof. Let $\{\gamma_\nu\}_{\nu \geq 1}$ be a minimizing sequence, i.e.

$$I(\gamma_\nu) \to \frac{1}{2}D \qquad \text{as } \nu \to \infty.$$

We write the metric tensor as $G := (g_{ij})_{i,j}$ and with this notation, we have

$$I(\gamma_\nu) := \frac{1}{2}\int_0^D \langle G(\gamma_\nu(t))\,\dot{\gamma_\nu}(t);\dot{\gamma_\nu}(t)\rangle\;\mathrm{d}t.$$

Since the metric tensor G is uniformly positive definite in $f(U)$, there exists a constant $\lambda > 0$ such that

$$\langle G(\gamma_\nu(t))\,\dot{\gamma_\nu}(t);\dot{\gamma_\nu}(t)\rangle \geq \lambda\,|\dot{\gamma_\nu}(t)|^2.$$

Thus, we have

$$\|\dot{\gamma_\nu}\|_{L^2([0,D];U)} \leq \frac{2}{\lambda} I(\gamma_\nu) \leq \frac{2}{\lambda} D.$$

Now since $\gamma_\nu - \gamma_0 \in W_0^{1,2}([0,D];U)$, by using Poincaré inequality, we have

$$\begin{aligned}
\|\gamma_\nu\|_{L^2([0,D];U)} &\leq \|\gamma_\nu - \gamma_0\|_{L^2([0,D];U)} + \|\gamma_0\|_{L^2([0,D];U)} \\
&\leq D\|\dot{\gamma_\nu} - \dot{\gamma_0}\|_{L^2([0,D];U)} + \|\gamma_0\|_{L^2([0,D];U)} \\
&\leq D\|\dot{\gamma_\nu}\|_{L^2([0,D];U)} + (D+1)\|\gamma_0\|_{W^{1,2}([0,D];U)} \\
&\leq \frac{2}{\lambda} D^2 + (D+1)\|\gamma_0\|_{W^{1,2}([0,D];U)}.
\end{aligned}$$

This implies $\{\gamma_\nu\}_{\nu \geq 1}$ is uniformly bounded in $W^{1,2}$ and thus, we deduce

$$\gamma_\nu \rightharpoonup \gamma \qquad \text{in } W^{1,2},$$

for some $\gamma \in \gamma_0 + W_0^{1,2}([0,D];U)$.

Note that here we have used the fact that $W_0^{1,2}$, being a convex subset of $W^{1,2}$, is weakly closed. However, here in dimension one, we could have also used the fact that

$$\gamma_\nu \rightharpoonup \gamma \quad \text{in } W^{1,2} \qquad \Rightarrow \qquad \gamma_\nu \to \gamma \quad \text{in } C^0.$$

Now we want to show that this weak limit γ is a minimizer. We have,

$$\begin{aligned}
I(\gamma_\nu) &= \frac{1}{2}\int_0^D \langle G\dot{\gamma_\nu}; \dot{\gamma_\nu}\rangle \\
&= \frac{1}{2}\int_0^D \langle G[\dot{\gamma} + (\dot{\gamma_\nu} - \dot{\gamma})]; \dot{\gamma} + (\dot{\gamma_\nu} - \dot{\gamma})\rangle \\
&= \frac{1}{2}\int_0^D \langle G\dot{\gamma}; \dot{\gamma}\rangle + \int_0^D \langle G\dot{\gamma}; \dot{\gamma_\nu} - \dot{\gamma}\rangle + \frac{1}{2}\int_0^D \langle G(\dot{\gamma_\nu} - \dot{\gamma}); \dot{\gamma_\nu} - \dot{\gamma}\rangle,
\end{aligned}$$

where we used the fact that G is symmetric. By the uniform positive definiteness of G, we have

$$\frac{1}{2}\int_0^D \langle G(\dot{\gamma_\nu} - \dot{\gamma}); \dot{\gamma_\nu} - \dot{\gamma}\rangle \geq 0.$$

Combining, we obtain

$$I(\gamma_\nu) \geq I(\gamma) + \int_0^D \langle G\dot{\gamma}; \dot{\gamma_\nu} - \dot{\gamma}\rangle.$$

Now since

$$\dot{\gamma_\nu} \rightharpoonup \dot{\gamma} \quad \text{in } L^2,$$

we deduce

$$\lim_{\nu \to \infty} \int_0^D \langle G\dot{\gamma}; \dot{\gamma_\nu} - \dot{\gamma} \rangle = 0.$$

Thus, we deduce

$$\frac{1}{2}D = \liminf_{\nu \to \infty} I(\gamma_\nu) \geq I(\gamma) + \lim_{\nu \to \infty} \int_0^D \langle G\dot{\gamma}; \dot{\gamma_\nu} - \dot{\gamma} \rangle = I(\gamma) \geq \frac{1}{2}D.$$

Hence γ is a minimizer. □

1.4 Regularity questions

1.4.1 Regularity of minimizers

Now we are going to show that this curve is actually C^2 and not just $W^{1,2}$. Results of this type are called regularity results. We show a general result.

Theorem 8 (Regularity)**.** *Let $f \in C^1\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $f = f(t, u, \xi)$ be such that*

- *f_ξ is C^1,*
- *$u \in W^{1,1}\left([a,b]; \mathbb{R}^N\right)$ is a critical point of the functional*

$$I[u] = \int_a^b f(t, u(t), \dot{u}(t)) \, \mathrm{d}t,$$

- *$f_u(t, u(t), \dot{u}(t))$, $f_\xi(t, u(t), \dot{u}(t))$ are L^1 and*
- *$f_{\xi\xi}$ is positive definite on $\Omega \times \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^{N+1}$ contains $\{(t, u(t)) : t \in [a,b]\}$.*

Then u is C^2. Moreover, if f_ξ is C^k for some $k \geq 2$, then u is C^{k+1}. In particular, u is C^∞ if f_ξ is C^∞.

Proof. We define the function $\phi : [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as

$$\phi(t, u, \xi, \eta) := f_\xi(t, u, \xi) - \eta.$$

Let $t_0 \in [a,b], u_0 = u(t_0)$, $\xi_0 = \dot{u}(t_0)$ and $\eta_0 = f_\xi(t_0, u(t_0), \dot{u}(t_0))$. Hence,

$$\phi(t_0, u_0, \xi_0, \eta_0) = 0.$$

Note that ϕ is C^1 since f_ξ is. Now as $f_{\xi\xi}$ is nonsingular, we can use implicit function theorem locally. Thus, we deduce that there exists an unique continuous function $\varphi = \varphi(t, u, \eta)$ which is C^1 and satisfies

$$\phi(t, u, \varphi(t, u, \eta), \eta) = 0 \tag{1.1}$$

in a neighborhood of $(t_0, u_0, \xi_0, \eta_0)$. However, since $(t, u(t), \dot{u}(t), f_\xi(t, u(t), \dot{u}(t)))$ also solves (1.1) in a neighborhood of $(t_0, u_0, \xi_0, \eta_0)$, we expect that by uniqueness, we shall have

$$\dot{u}(t) = \varphi(t, u(t), f_\xi(t, u(t), \dot{u}(t))).$$

However, we can not claim it just yet. The uniqueness conclusion of implicit function theorem holds for continuous functions and we do not yet know if $\dot{u}$ is continuous. So we need to prove the uniqueness differently in a larger class.

Suppose for a given $(t, u, q) \in \Omega \times \mathbb{R}^N$, there exist two solutions $p_1, p_2 \in \mathbb{R}^N$ such that

$$q = f_\xi(t, u, p_1) \quad \text{and} \quad q = f_\xi(t, u, p_2).$$

Thus, we have

$$\int_a^b [f_{\xi\xi}(t, u, sp_1 + (1-s)p_2)](p_2 - p_1)\ \mathrm{d}s = 0.$$

Since $f_{\xi\xi}$ is positive definite, this implies $p_1 = p_2$.

The uniqueness we just proved implies

$$\dot{u}(t) = \varphi(t, u(t), f_\xi(t, u(t), \dot{u}(t))) \tag{1.2}$$

for almost all t in a neighborhood of t_0. Now, $u(t)$ is absolutely continuous w.r.t. t. We can also prove $f_\xi(t, u(t), \dot{u}(t))$ is absolutely continuous w.r.t t since u is a critical point. So the RHS of (1.2) is an absolutely continuous function, say $v(t)$. But if $\dot{u}$ agrees with an absolutely continuous function for a.e. t in a neighborhood of t_0, we have

$$u(t) = u(t_0) + \int_{t_0}^t \dot{u}(s)\ \mathrm{d}s = u(t_0) + \int_{t_0}^t v(s)\ \mathrm{d}s,$$

for a.e. t in a neighborhood of t_0. The LHS above is clearly C^1, hence so is u and thus $\dot{u}$ is continuous. So now the uniqueness for implicit function theorem implies (1.2) holds and u is C^2. $\square$