Introduction to the Calculus of Variations
Lecture Notes
Lecture 10

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Chapter 1

Prelude to Direct Methods

1.1 Geodesics: the problem

1.2 Absolute continuity: first encounter with Sobolev spaces

Recap We have already defined weak derivatives.

Definition 1 (weak derivatives)**.** *Let $u \in L^1\left((0,T);\mathbb{R}^d\right)$. We say u has a* ***weak derivative*** *if there exists a function $v \in L^1\left((0,T);\mathbb{R}^d\right)$ such that*

$$\boxed{\int_0^T \langle v, \psi\rangle = -\int_0^T \left\langle u, \dot{\psi}\right\rangle \qquad \text{for any } \psi \in C_c^\infty\left((0,T);\mathbb{R}^d\right).}$$

In this case, we say v is the weak derivative of u and we write

$$v = \dot{u}.$$

Remark 2. *The weak derivative, if it exists, is unique.*

Can you see why? Any two weak derivatives of u would be equal a.e. by the fundamental lemma of calculus of variations and thus would represent the same L^1 function.

1.2.1 Sobolev spaces in dimension one: definition and elementary properties

Definition 3 ($W^{1,p}$ functions)**.** *A measurable function $u : (a,b) \to \mathbb{R}$ is said to be a* ***Sobolev function*** *of class $W^{1,p}$ if $u \in L^p((a,b))$ and the weak derivative $\dot{u} \in L^p((a,b))$ for $1 \le p \le \infty$. In this case, we write $u \in W^{1,p}((a,b))$.*

A measurable function $u : (a,b) \to \mathbb{R}^N$ is said to be a ***Sobolev function*** *of class $W^{1,p}$ if $u_i \in W^{1,p}((a,b))$ for every $1 \le i \le N$.*

Remark 4. *Note that by our definition, as soon as an L^1 function is weakly differentiable, it is a Sobolev function of class $W^{1,1}$.*

Let us now introduce a norm on $W^{1,p}$.

Proposition 5. *Let $u \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$. If $1 \le p < \infty$, then*

$$\boxed{\|u\|_{W^{1,p}((a,b);\mathbb{R}^N)} := \|u\|_{L^p((a,b);\mathbb{R}^N)} + \|\dot{u}\|_{L^p((a,b);\mathbb{R}^N)} < \infty.}$$

For $p = \infty$, we have

$$\boxed{\|u\|_{W^{1,\infty}((a,b);\mathbb{R}^N)} := \|u\|_{L^\infty((a,b)\mathbb{R}^N)} + \|\dot{u}\|_{L^\infty((a,b);\mathbb{R}^N)} < \infty.}$$

Moreover, these expressions defines a norm on the vector space of all functions in $W^{1,p}\left((a,b);\mathbb{R}^N\right)$.

Proposition 6. *The vector space of all function in $W^{1,p}\left((a,b);\mathbb{R}^N\right)$, equipped with the norms above is a* ***Banach space****, which is reflexive for $1 < p < \infty$ and is separable for $1 \le p < \infty$. We would simply write this space as $W^{1,p}\left((a,b);\mathbb{R}^N\right)$.*

Proposition 7. *The space $W^{1,2}\left((a,b);\mathbb{R}^N\right)$, equipped with the inner product*

$$\boxed{\begin{aligned}\langle u,v\rangle_{W^{1,2}((a,b);\mathbb{R}^N)} &:= \langle u,v\rangle_{L^2((a,b);\mathbb{R}^N)} + \langle \dot{u},\dot{v}\rangle_{L^2((a,b);\mathbb{R}^N)} \\ &= \int_a^b \langle u,v\rangle + \int_a^b \langle \dot{u},\dot{v}\rangle ,\end{aligned}}$$

is a ***Hilbert space****.*

There is another way the Sobolev spaces could have been defined for $1 \le p < \infty$.

Definition 8 (Sobolev spaces $H^{1,p}$)**.** *Let $X^{1,p}$ be the linear subspace of $C^1\left((a,b);\mathbb{R}^N\right)$ functions such that*

$$\boxed{\|u\|_{W^{1,p}((a,b);\mathbb{R}^N)} := \|u\|_{L^p((a,b);\mathbb{R}^N)} + \|\dot{u}\|_{L^p((a,b);\mathbb{R}^N)} < \infty.}$$

The completion of $X^{1,p}$ with respect to the above norm is called $H^{1,p}\left((a,b);\mathbb{R}^N\right)$.

Extension and density results: $W^{1,p} = H^{1,p}$

We are now going to prove that the two spaces $W^{1,p}$ and $H^{1,p}$ are the same. In particular, we prove smooth functions are dense in $W^{1,p}$. To show this, we shall also prove that any function $u \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$ is actually the restriction of a $W^{1,p}\left(\mathbb{R};\mathbb{R}^N\right)$ function.

Theorem 9 (extension and density)**.** *Let (a,b) be a bounded interval of $\mathbb{R}$ and let $u \in W^{1,p}((a,b))$ with $1 \le p < \infty$. Then*

1. *There exists a function $U \in L^p(\mathbb{R})$ which has a weak derivative $\dot{U} \in L^p(\mathbb{R})$ and satisfies $U = u$ in (a, b).*

2. $u \in H^{1,p}((a, b))$.

Proof. We first prove the first part. Pick $\bar{a}, \bar{b} \in \mathbb{R}$ with $a < \bar{a} < \bar{b} < b$ and let $\eta \in C^1(\mathbb{R})$ be such that

$$\eta = 1 \text{ in } (-\infty, \bar{a}) \qquad \text{and} \qquad \eta = 0 \text{ in } (\bar{b}, \infty).$$

Our plan is to write

$$u = \eta u + (1 - \eta) u.$$

We can check that

$$\eta u \in W^{1,p}((a, \infty)) \quad \text{and} \quad (1 - \eta) u \in W^{1,p}((-\infty, b)).$$

Now we define

$$U_1(t) = \begin{cases} [\eta u](t), & t > a \\ [\eta u](2a - t), & t < a \end{cases} \quad \text{and} \quad U_2 = \begin{cases} [(1 - \eta) u](t), & t < b \\ [(1 - \eta) u](2b - t), & t > b. \end{cases}$$

Clearly, $U = U_1 + U_2$ does the job.

Now let us prove the second part. Let $U \in W^{1,p}(\mathbb{R})$ be the above extension of $u \in W^{1,p}((a, b))$. Pick a nonnegative $\phi \in C_c^\infty([-1, 1])$ such that $\int \phi = 1$ and set

$$\phi_\varepsilon(t) := \frac{1}{\varepsilon} \phi\left(\frac{t}{\varepsilon}\right).$$

Then we can easily check that

$$U_\varepsilon := U * \phi_\varepsilon$$

is smooth and converges to U in the $W^{1,p}$ norm on $\mathbb{R}$. □

Now we want to investigate the question of boundary values (or any pointwise value) of a $W^{1,p}$ function. Note since $W^{1,p}$ functions are only a priory L^p functions, they are only defined a.e. and thus the pointwise value does not necessarily make sense! Later, we would resolve this issue by the **trace map**. In one dimension, however, we are in luck. As we show now, these functions are actually **continuous** in one dimension.

Continuity of Sobolev functions in one dimension

Theorem 10 (Continuity of $W^{1,1}$ functions in one dimension)**.** *Every function in $W^{1,1}((a, b))$ is uniformly continuous in $[a, b]$. In particular,*

$$W^{1,1}((a, b)) \subset C^0([a, b])$$

and

$$\sup_{t\in[a,b]} |u| \leq \frac{1}{(b-a)} \int_a^b |u| + \int_a^b |\dot{u}| \,.$$

Moreover, the fundamental theorem of calculus holds, i.e. for all $a \leq s < t \leq b$,

$$u(t) - u(s) = \int_s^t \dot{u}(\theta)\ \mathrm{d}\theta.$$

This is something we have already seen implicitly in attempting to solve the geodesic problem before.

Proof. Since $W^{1,1} = H^{1,1}$, for $u \in W^{1,1}((a,b))$, there exists a sequence $\{u_\nu\}_{\nu\geq 1} \subset X^{1,1}$ such that

$$u_\nu \to u \qquad \text{in } W^{1,1}.$$

Now, using the fundamental theorem of calculus, we obtain

$$\boxed{u_\nu(t) - u_\nu(s) = \int_s^t \dot{u_\nu}(t)\ \mathrm{d}t.} \tag{1.1}$$

Thus, in particular, we have,

$$\boxed{|u_\nu(t) - u_\nu(s)| = \left|\int_s^t \dot{u_\nu}(t)\ \mathrm{d}t\right| \leq \int_s^t |\dot{u_\nu}(t)|\ \mathrm{d}t.}$$

and

$$\boxed{|u_\nu(t)| \leq |u_\nu(s)| + \int_s^t |\dot{u_\nu}(t)|\ \mathrm{d}t.}$$

The last inequality implies

$$\boxed{|u_\nu(t)| \leq |u_\nu(s)| + \int_a^b |\dot{u_\nu}(t)|\ \mathrm{d}t.}$$

Integrating this with respect to $s \in (a,b)$, we obtain

$$\boxed{|u_\nu(t)| \leq \frac{1}{(b-a)} \int_a^b |u_\nu(s)|\ \mathrm{d}s + \int_a^b |\dot{u_\nu}(t)|\ \mathrm{d}t.} \tag{1.2}$$

Thus $\{u_\nu\}$ is uniformly bounded in C^0 and as

$$\dot{u_\nu} \to \dot{u} \text{ strongly in } L^1,$$

we have

$$\int_s^t |\dot{u_\nu}(t)|\ \mathrm{d}t \to 0 \qquad \text{uniformly in } \nu \qquad \text{as } t - s \to 0.$$

Indeed, since $\dot{u} \in L^1$, we have

$$\int_s^t |\dot{u}(t)| \, \mathrm{d}t \to 0 \qquad \text{as } t - s \to 0.$$

But the strong convergence implies

$$\int_s^t |\dot{u_\nu}(t) - \dot{u}(t)| \, \mathrm{d}t \to 0 \qquad \text{as } \nu \to 0.$$

These two together implies the claim above.

But the inequality

$$\boxed{|u_\nu(t) - u_\nu(s)| = \left| \int_s^t \dot{u_\nu}(t) \, \mathrm{d}t \right| \leq \int_s^t |\dot{u_\nu}(t)| \, \mathrm{d}t.} \tag{1.3}$$

together with the fact that

$$\int_s^t |\dot{u_\nu}(t)| \, \mathrm{d}t \to 0 \quad \text{uniformly in } \nu \quad \text{as } t - s \to 0,$$

implies

$$|u_\nu(t) - u_\nu(s)| \to 0 \quad \text{uniformly in } \nu \quad \text{as } t - s \to 0.$$

This implies that $\{u_\nu\}$ is equicontinuous and thus by Ascoli-Arzela theorem, up to the extraction of a subsequence which we do not relabel, we have

$$u_\nu \to u \qquad \text{in } C^0.$$

This shows u is continuous. Now, passing to the limit in (1.3), we deduce that u is **uniformly continuous**. The other statements follow by passing to the limit in (1.1) and (1.2). □

In a similar manner, we can prove the following, which is a particular case of the Sobolev-Morrey embedding.

Theorem 11 (Continuity of $W^{1,p}$ functions in one dimension)**.** *Every function in $W^{1,p}((a,b))$ with $p > 1$ Hölder continuous in $[a,b]$. In particular,*

$$W^{1,p}((a,b)) \subset C^{0,1-\frac{1}{p}}([a,b])$$

and

$$\sup_{t \in [a,b]} |u| \leq \left(\frac{1}{(b-a)} \int_a^b |u|^p \right)^{\frac{1}{p}} + (b-a)^{1-\frac{1}{p}} \left(\int_a^b |\dot{u}|^p \right)^{\frac{1}{p}}.$$

Moreoever for all $s, t \in [a,b]$, we have,

$$|u(t) - u(s)| \leq \left(\int_a^b |\dot{u}|^p \right)^{\frac{1}{p}} |t - s|^{1-\frac{1}{p}}.$$

Proof. The proof is almost the same. The only step where it differs is that we now need to apply Hölder inequality to

$$\boxed{|u_\nu(t) - u_\nu(s)| = \left|\int_s^t \dot{u_\nu}(t)\ \mathrm{d}t\right| \le \int_s^t |\dot{u_\nu}(t)|\ \mathrm{d}t.}$$

to deduce

$$\boxed{\begin{aligned} |u_\nu(t) - u_\nu(s)| &\le \int_s^t |\dot{u_\nu}(t)|\ \mathrm{d}t \\ &\le \left(\int_s^t |\dot{u}|^p\right)^{\frac{1}{p}} |t-s|^{1-\frac{1}{p}} \\ &\le \left(\int_a^b |\dot{u}|^p\right)^{\frac{1}{p}} |t-s|^{1-\frac{1}{p}}. \end{aligned}}$$

The rest is the same. □

1.2.2 Functions with zero boundary values in $W^{1,p}$ in one dimension

Now we are going to characterize the functions with zero boundary values.

Definition 12 ($W_0^{1,p}$)**.** *We define the space* $W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$ *as the completion of*

$$X_0^{1,p} := \left\{u \in C_c^\infty\left((a,b);\mathbb{R}^N\right) : \|u\|_{W^{1,p}((a,b);\mathbb{R}^N)} < \infty\right\}$$

with respect to the $W^{1,p}$ *norm.*

Clearly, if $u \in W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$, then $u(a) = 0 = u(b)$. We can prove the converse as well.

Theorem 13 (Characterization of $W_0^{1,p}$)**.** *Let* $u \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$. *Then* $u \in W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$ *if and only if* $u(a) = 0 = u(b)$.

Proof. Fix any function $G \in C^1(\mathbb{R})$ such that

$$G(t) = \begin{cases} 0 & \text{if } |t| \le 1, \\ t & \text{if } |t| \ge 2. \end{cases}$$

and

$$|G(t)| \le |t| \quad \text{for all } t \in \mathbb{R}.$$

Set

$$u_\nu = \frac{1}{\nu} G(\nu u),$$

so that $u_\nu \in W^{1,p}\left((a,b);\mathbb{R}^N\right)$. On the other hand, we can check that the support of u_ν is compactly contained in (a,b) since $u\,(a) = 0 = u\,(b)$ and u is continuous. But this implies easily that $u_\nu \in W_0^{1,p}\left((a,b);\mathbb{R}^N\right)$. Finally, one easily checks that

$$u_\nu \to u \qquad \text{in } W^{1,p}\left((a,b);\mathbb{R}^N\right)$$

by the dominated convergence theorem. □