Introduction to the Calculus of Variations
Lecture Notes
Lecture 1

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Introduction

Finding the minima of a function

We begin with the study of a considerably simpler problem, which we all learned how to solve in our Calculus courses - finding a minima of a given function. However, this would already give us a glimpse of what is to come.

Finding the minima of a function Let $X \subset \mathbb{R}^n$ and let $F : X \to \mathbb{R}$ be a given function. Consider the minimization problem

$$m = \inf\{F(x) : x \in X\} \tag{P}$$

If the problem can be solved at all, there are roughly two methods to solve the problem. The methods are called **the classical method** and **the direct method**.

Classical Method

Assume F is C^1. Then $\bar{x} \in X$ is a **stationary point** or **critical point**[1] of F if and only if it solves the equation

$$\nabla F(\bar{x}) = 0. \tag{1}$$

This equation is our first instance of what are known as **Euler-Lagrange Equations**. Roughly speaking, the Euler-Lagrange equation is the equation satisfied by all critical points of the function. In the Calculus of Variations, we would be interested in finding a minimizer for a functional, rather than in finding a minima of a function. But the philosophy stays the same. The Euler-Lagrange equation associated to a functional is the equation all critical points of the functional must satisfy.

[1] We would use the phrases 'stationary points' and 'critical points' interchangeably, since we have already assumed F to be differentiable. If F is not assumed to be differentiable, then technically speaking, there is a slight difference between the two notions. The notion of critical point includes stationary points, but also includes points where F is not differentiable. On the other hand, a stationary point is where F is *differentiable and the gradient is zero.*

The classical method consists of trying to solve the Euler-Lagrange equations to obtain all critical points and then checking if one (or several) of them is a local minima.

To check if a given critical point $\bar{x} \in X$ is a minima, one checks if we have

$$F(\bar{x}) \leq F(y) \qquad \text{for all } y \in B_r(\bar{x})$$

for some $r > 0$. If F is C^2, then a sufficient criterion, which is much simpler to check in practice, is the following. If F is C^2, then a given critical point $\bar{x} \in X$ is a minima if

$$\nabla^2 F(\bar{x}) > 0, \quad \text{i.e. the Hessian at } \bar{x} \text{ is positive definite.} \tag{2}$$

Remark 1. *Some authors call this 'strictly positive definite' and denote the non-strict inequality case as 'positive definite'. We would use the terms positive definite and positive semidefinite respectively.*

This is exactly how we found minima in Calculus courses! Note that if F is strictly convex and C^2, (2) is automatic. The minima would also be unique in that case. The classical picture to keep in mind is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2,$$

which is convex, C^2 (indeed, C^∞) and the origin is the unique global minima. However, if F is **convex** and C^2, but not necessarily strictly convex, then **any critical point $\bar{x}$ is a local minima**. Such a minima would also be a global minima, i.e. the value of F at such a minima is globally the minimum value of F. However, uniqueness of minima does not necessarily hold. Try to think of an example and see Assignment 1.

Convexity in general plays an important role in minization problems. Though we are calculating first derivatives of the function and convexity might appear to be a secondary issue, we shall see soon that convexity is in fact, much more crucial for minimization problems than differentiability!

Direct Method

Although the method in the previous subsection is probably the only one you have encountered in your calculus courses, there is another method. Let us first describe the crux of the method.

Let $\{x_s\} \subset X$ be a **minimizing sequence** for (P), i.e.

$$\lim_{s \to \infty} F(x_s) = \inf\{F(x) : x \in X\}. \tag{3}$$

Such sequences must exist, by definition of the infimum! Now suppose the hypotheses on F allow us to prove sequential compactness of all such minimizing sequences. Typically, we can deduce an uniform bound

$$\|x_s\| \leq C \qquad \text{for all } s.$$

Then up to the extraction of a subsequence that we do not relabel, we get

$$x_s \to \bar{x}, \quad \text{for some } \bar{x} \in X. \quad \text{(by Bolzano-Weierstrass theorem)} \tag{4}$$

Now if F is lower semicontinuous, we obtain

$$\inf\{F(x) : x \in X\} \overset{\text{since } \bar{x} \in X}{\leq} F(\bar{x}) \overset{\text{lsc}}{\leq} \lim_{s \to \infty} F(x_s) \overset{\text{min seq}}{=} \inf\{F(x) : x \in X\}.$$

Thus all inequalities must be equalities and $\bar{x}$ is a minima.

What we have shown above is an *abstract existence theorem*, i.e. we have proved the existence of a minima for a function satisfying some condition, namely sequential compactness of minimizing sequences and lower semicontinuity. Just as a remark, one can prove that a **convex function is continuous** and thus automatically lower semicontinuous as well. Obtaining the uniform bound for minimizing sequences is not always easy. But these generally follow from **coercivity** assumptions on F. Let us first define **coercivity**. We allow extended real valued functions as well. The usual definition is as follows.

Definition 2. *A function $F : \mathbb{R}^n \to \mathbb{R} \cap \{-\infty, +\infty\}$ is said to be* coercive *if we have*

$$F(x) \to +\infty \quad \text{as } \|x\| \to +\infty.$$

However, this definition clearly works only for extended real valued functions. For our purposes, we would use another definition, which is weaker and makes sense even for functions $F : \mathbb{R}^n \to \mathbb{R}^N$ (in fact even for maps between two normed linear spaces, in particular, Banach spaces). This is usually called **norm-coercivity** and is strictly weaker than coercivity for real valued functions (think of an real-valued function which is norm-coercive but not coercive), but we would refer to it coercivity.

Definition 3 (Coercivity or Norm-Coercivity)**.** *A function $F : \mathbb{R}^n \to \mathbb{R}^N$ is said to be* norm-coercive *or simply* coercive *if we have*

$$\|F(x)\| \to +\infty \quad \text{as } \|x\| \to +\infty.$$

Now let us show that for function which are coercive and bounded below, minimizing sequences must be uniformly bounded.

Proposition 4. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be coercive and bounded below, i.e.*

$$\inf\{F(x) : x \in \mathbb{R}^n\} > -\infty.$$

Then any minimizing sequence for F is uniformly bounded.

Proof. Let $\{x_s\} \subset \mathbb{R}^n$ be a minimizing sequence for F, i.e.

$$\lim_{s \to \infty} F(x_s) = \inf\{F(x) : x \in \mathbb{R}^n\}. \tag{5}$$

If there exist no constant $C > 0$ such that $\|x_s\| \leq C$ for all s, then there must exist a subsequence, which we do not relabel, such that

$$\|x_s\| \to +\infty.$$

Since F is coercive, this implies

$$\|F(x_s)\| \to +\infty.$$

But this contradicts (5). □

There are many conditions which imply coercivity of F. For example, $F : \mathbb{R}^n \to \mathbb{R}$ is said to have **superlinear growth at infinity** if it satisfies

$$\lim_{\|x\| \to \infty} \frac{|F(x)|}{\|x\|} = +\infty.$$

Clearly, this implies coercivity and thus implies the uniform bound for minimizing sequences. (Check!)

A typical example: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = |x|.$$

A glance at the graph of this function is enough to convince you that the function has a unique, global minima at the origin. In fact, it is fairly easy to establish the existence of a minima for this function by direct methods (Try it!). Note that the function is not differentiable at the origin, so it is not clear how to use the classical methods (However, it can be done. Think about it!)

At this point, the direct method may not look like much. In Assignment 1, there are a number of results about the existence of minima using the direct methods which might convince you otherwise.

Finding minima: Comparison of the methods

The classical method

- works by solving the EL equations
- finds all critical points
- needs F to be reasonably regular

The direct method

- works directly with the function, not the Euler-Lagrange equations (Hence the name!)
- finds only minima, not all critical points (There are direct methods for finding other critical points too, but those are beyond the scope of this course)

- F can be less regular, but has to be coercive and lower semicontinuous.

The basic features of the two methods are the same in the calculus of variations too. But there are some differences as well. For finding minima of a function, the EL equations are algebraic equations. In the Calculus of Variations, the EL equations are ODE, system of ODEs, PDE or a system of PDEs.

If they are PDEs or system of PDEs, even proving existence of a solution (let alone characterizing all solutions!) directly is hard! In fact, PDE theory is one of the main applications of the Calculus of Variations! Conversely, variational methods are among the most important tools in PDE theory. If a PDE appears as the EL equation of some functional (which by the way is often the case), we usually prove existence of a solution by finding a critical point for the functional by direct methods, precisely going in the reverse direction as compared to the classical methods.

Function to functionals

Calculus of Variations: The abstract problem

We now want to pass from functions to functionals. Let us state our model problem.

Let $\Omega \subset \mathbb{R}^n$ open, bounded, smooth. $n, N \geq 1$ are integers. Let $\mathcal{A}$ be a given class of functions $u : \Omega \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a given function. Consider the following minimization problem

$$\inf \left\{ I(u) := \int_\Omega f\left(x, u(x), Du(x)\right) \, \mathrm{d}x : u \in \mathcal{A} \right\} \tag{P}$$

- The integral functional $I(u)$ is called the **Lagrangian**
- The integral f is called the **Lagrangian density**
- The class $\mathcal{A}$ is called the **class of admissible functions**.

The Lagrangian can depend on higher order derivatives of u. Those however are somewhat rare, though notable exceptions exist (e.g. Polyharmonic maps).

History of the Calculus of Variations and classical problems

A brief history of the Calculus of Variations

Antiquity Perhaps the oldest known problem in the calculus of variations is the **isoperimetric problem**, which is just the **isoperimetric inequality** in

dimension two.
The problem is to find *the(?)* geometric figure which has **the largest area with a fixed perimeter**.

The fact that the **circle** has this property is probably known since antiquity in many cultures around the world, including Greece, Egypt, India, Babylon, China etc. In Europe, it was traditionally known as the Dido problem. Around 200 BCE, Zenodorous proved the inequality for polygons. A number of prominent names worked on the problems, including the likes of **Archimedes**, **Pappus**, **Euler**, **Galileo**, **Legendre**, **Riccati**, **Steiner** etc. The first proof that agrees with modern standards is due to **Weierstrass**. Later improvements and refinements are due to **Blaschke**, **Carathodory**, **Frobenius**, **Hurwitz**, **Lebesgue**, **Liebmann**, **Minkowski**, **H.A. Schwarz**, **Sturm**, **Tonelli** among others.

Seventeenth century and the Brachistochrone Seventeenth century witnessed the rapid rise to prominence of the calculus of variations. Several problems was posed and solutions were attempted. Also, from this time onwards, people started realising the deep connection of physics and engineering with the calculus of variations. The central problems that were posed and studied in this era includes

- **Fermat (1662)** - geometric optics,
- **Newton (1685)** and **Huygens (1691)** - bodies moving through a fluid,
- **Gallileo (1638)** formulated the **Brachistochrone problem**, solved by **John Bernoulli (1696)**, **James Bernoulli**, **Newton** and **Leibnitz**.

A significant breakthrough of the subject was achieved when **Euler** and **Lagrange** introduced what is now known as the **Euler-Lagrange equation**. Throughout this period, the works of **Bliss**, **Bolza**, **Carathodory**, **Clebsch**, **Hahn**, **Hamilton**, **Hilbert**, **Kneser**, **Jacobi**, **Legendre**, **Mayer**, **Weierstrass** and many many others, deepened our understanding of the subject.

Nineteenth century and the Dirichlet integral The Dirichlet integral and the associated minimzation problems were introduced and several attempts were made to solve it, most notably by **Dirichlet**, **Gauss**, **Thompson** and **Riemann**. However, those attempts remained unsuccessful. Finally **Hilbert** solved the problem, extending significant and breakthrough contributions of **Lebesgue** and **Tonelli**.

This problem *inspired the development of most of modern analysis, namely functional analysis, measure theory, distribution theory, Sobolev spaces, partial differential equations.*

Minimal surfaces: Seventeenth to Twentieth Century This is another central problem which has inspired a lot of analysis, including subjects like **geometric measure theory**. **Lagrange** first formulated the problem in 1762.

However, this is often called **Plateau's problem** in honor of the Belgian physicist **Joseph Plateau**, whose **experiments with soap films** and his empirical 'Plateau's laws' influenced the status of the problem considerably.

Names such as **Ampre**, **Beltrami**, **Bernstein**, **Bonnet**, **Catalan**, **Darboux**, **Enneper**, **Haar**, **Korn**, **Legendre**, **Lie**, **Meusnier**, **Monge**, **Mntz**, **Riemann**, **H.A. Schwarz**, **Serret**, **Weierstrass**, **Weingarten** and others worked on the problem.

Douglas and **Rado** finally solved the problem in 1930. Douglas was awarded the *fields medal* for it in 1936! Later contributions and further refinements are due to **Courant**, **Leray**, **Mac Shane**, **Morrey**, **Morse**, **Tonelli** etc.

Classical examples

Now we are going to give a few classical examples, almost all of which were instrumental in driving the early research in the calculus of variations and paved the way for later developments.

- **Fermat's principle of least time**: The basic variational principle in geometric optics.
- **Newton's problem**: Finding the surface of revolution which experiences least resistance when moving through a fluid.
- **Brachistochrone**: Almost *the* iconic example in the classical calculus of variations.
- **Principle of least action**: Essentially the heart and soul of Newtonian mechanics.
- **Minimal surface of revolution** : The easier version of another iconic example: the minimal surface problem
- **Dirichlet integral**: The most celebrated and the protypical example in all the calculus of variations.
- **Minimal surfaces**: Another star of the show! Almost as famous as the Dirichlet integral.
- **Isoperimetric inequality**

Fermat's principle of least time: Find the path of a light ray in a medium with nonconstant refractive index.
The ray follows the path of least time!
The variational problem:

$$\inf\left\{I(u) := \int_a^b f\left(x, u(x), u'(x)\right)\ \mathrm{d}x : u(a) = \alpha, u(b) = \beta\right\}, \tag{6}$$

where $n = N = 1$ and the form of the Lagrangian is

$$f(x, u, \xi) = g(x, u)\sqrt{1 + \xi^2}.$$

Newton's optimal surface of revolution with least fluid resistance: Find the surface of revolution that experiences the least resistance while moving through a fluid.
The variational problem:

$$m = \inf\left\{I(u) := \int_a^b f\left(u(x), u'(x)\right)\ \mathrm{d}x : u(a) = \alpha, u(b) = \beta\right\}, \tag{7}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = 2\pi u\left(\frac{\xi^3}{1 + \xi^2}\right).$$

Brachistochrone: Find the quickest path between two points for a point mass moving under gravity.
Let one of the points be the origin $(0, 0) \subset \mathbb{R}^2$ and the other point is $(b, -\beta) \subset \mathbb{R}^2$ with $b, \beta > 0$. Gravity is acting downwards in the negative y-axis and the path is expressed as $(x, -u(x))$ with $0 \leq x \leq b$.

The variational problem:

$$m = \inf\left\{I(u) := \int_a^b f\left(u(x), u'(x)\right)\ \mathrm{d}x : u \in \mathcal{A}\right\}, \tag{8}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = \sqrt{\left(\frac{1 + \xi^2}{2gu}\right)}.$$

The class of admissible paths is

$$\mathcal{A} := \left\{\begin{array}{c} u \in C^1\left([0, b]\right) : u(0) = 0,\ u(b) = \beta \\ \text{and } u(x) > 0 \text{ for all } x \in (0, b] \end{array}\right\}.$$

The solution is called a **Cycloid**, the curve traced by a point on the rim of a rolling (without slipping) wheel. This curve also has another remarkable property. The time needed for the particle to slide along the cycloid to the final point $(b, -\beta)$ is the same for any initial point (x, y) on the cycloid, not just $(0, 0)$! For this property, the cycloid is also called a **Tautochrone**.

Principle of least action - mechanics of system of point masses: Find the configuration of M point masses moving under a potential at time T.
Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \leq i \leq M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t. The **potential energy** function for the configuration $u(t)$ is a given function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$.

The variational problem:

$$m = \inf \left\{ I(u) := \int_0^T f\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t : u(0) = u_0, \dot{u}(0) = v_0 \right\}, \qquad (9)$$

where $n = 1, N = 3M$, u_0, v_0 given and the form of the Lagrangian density is

$$f(x, u, \xi) = T\left(\xi\right) - U\left(t, u(t)\right). \qquad (\text{ usually called } \textbf{action} \text{ })$$

Here T is the **kinetic energy** and is given by

$$T\left(\xi\right) := \frac{1}{2} \sum_{i=1}^{M} m_i \xi_i^2.$$

This variational problem can be justifiably called the heart and soul of Newtonian mechanics. In the next chapter, when we would be discussing Euler-Lagrange equations, we shall see that the Euler-Lagrange equation associated to this variational problem is nothing but the familiar **Newton's laws of motion**!

Minimal surface of revolution: Determine the one with minimal area among all surfaces of revolution of the form

$$v(x, y) = (x, u(x) \cos y, u(x) \sin y)$$

with fixed end points $u(a) = \alpha, u(b) = \beta$.

Here $n = N = 1$ and the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = 2\pi u \sqrt{1 + \xi^2}$$

and the variational problem is

$$\inf \left\{ I(u) = \int_a^b f(u(x), u'(x)) \, \mathrm{d}x : u(a) = \alpha, u(b) = \beta, u > 0 \right\} = m.$$

The solutions are called **Catenoids**, which is the surface of revolution obtained by rotating a **Catenary** curve about its directrix. Euler in 1744 discovered them and proved them to be minimal surfaces. Apart from the plane, they are the only other minimal surfaces of revolutions. They can also be thought of as special cases of a larger family of minimal surfaces (not necessarily surfaces of revolutions) called the **Helicoids**.

Dirichlet integral: Arguably the most celebrated problem in all of the calculus of variations. We have here $n > 1, N = 1$ and

$$\inf \left\{ I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \ \mathrm{d}x : u = u_0 \text{ on } \partial\Omega \right\} = m.$$

The Euler-Lagrange equation is nothing other than the **Laplace equation**, namely

$$\Delta u = 0.$$

A generalized version of this is the p**-Dirichlet integral**,

$$\inf \left\{ I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \ \mathrm{d}x : u = u_0 \text{ on } \partial\Omega \right\} = m,$$

where $1 < p < \infty$. The Euler-Lagrange equation is the p**-Laplace equation**, i.e.

$$\Delta_p u := \operatorname{div} \left(|\nabla u|^{(p-2)} \nabla u \right) = 0.$$

This is quasilinear if $p \neq 2$, degenerate elliptic if $p > 2$ and singular elliptic if $1 < p < 2$. There are also vectorial versions of both the problems, where $N \geq 2$, whose EL equations are called Laplace system (more usually, it is still called Laplace equation though) and p-Laplacian system.

Harmonic and p-harmonic maps: There are vectorial versions of the Dirichlet and p-Dirichlet integral which are different (and much harder to understand) from the usual vectorial versions. Introducing them requires some knowledge of geometry. Just very briefly, we consider apparently the same minimization problem (since the differential du is exactly what the gradient would be in the $\mathbb{R}^N$ case)

$$\inf \left\{ I(u) = \frac{1}{2} \int_\Omega |du|^2 \ \mathrm{d}x : u = u_0 \text{ on } \partial\Omega \right\} = m.$$

However, we require that u takes values in a Riemannian minifold N. That is the Dirichlet integral for manifold-valued maps. One can also consider the 'same' minimization problem for maps $u : M \to N$, where M and N are both Riemannian manifolds. The minimizers are called harmonic maps between the manifolds. Although the minimization problem apparently looks the same, there is a very real difference from the usual vectorial version. The norm of the differential is calculated using the metric of M and N, which would reduce to the usual vectorial version when M and N are taken as $\bar{\Omega} \subset \mathbb{R}^n$ and $\mathbb{R}^N$ respectively. p-Harmonic maps between manifolds can be defined in a similar manner.

Minimal surfaces: The question is to find among all surfaces $\Sigma \subset \mathbb{R}^3$ (or more generally in $\mathbb{R}^{n+1}$, $n \geq 2$) with prescribed boundary, $\partial\Sigma = \Gamma$, where Γ is a simple closed curve, one that is of minimal area.

A variant of this problem is known as *Plateau problem.*

One can experimentally realize such surfaces by dipping a wire loop into soapy water; the surface obtained when pulling the wire out from the water is a minimal surface.

The precise formulation of the problem depends on the kind of surfaces that we are considering. We have seen above how to write the problem for minimal surfaces of revolution. We now formulate the problem for more general surfaces.

Minimal surfaces: Nonparametric surfaces We consider (hyper) surfaces of the form

$$\Sigma = \left\{ v(x) = (x, u(x)) \in \mathbb{R}^{n+1} : x \in \overline{\Omega} \right\}$$

with $u : \overline{\Omega} \to \mathbb{R}$ and where $\Omega \subset \mathbb{R}^n$ is a bounded connected open set.

These surfaces are therefore graphs of functions.

The fact that $\partial\Sigma$ is a preassigned curve, Γ, reads now as $u = u_0$ on $\partial\Omega$, where u_0 is a given function. The area of such a surface is given by

$$\text{Area}(\Sigma) = I(u) = \int_\Omega f(\nabla u(x))\, dx$$

where, for $\xi \in \mathbb{R}^n$, we have set

$$f(\xi) = \sqrt{1 + |\xi|^2}\,.$$

The problem is then written in the usual form

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(\nabla u(x))\, dx : u = u_0 \text{ on } \partial\Omega \right\},$$

with

$$f(\xi) = \sqrt{1 + |\xi|^2} \qquad \text{for } \xi \in \mathbb{R}^n.$$

Associated with (P) we have the so-called *minimal surface equation*

$$(E) \quad \mathrm{M}u \equiv \left(1 + |\nabla u|^2\right) \Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

which is the equation that any minimizer u of (P) should satisfy. In geometrical terms, this equation just expresses the fact that the corresponding surface Σ has everywhere vanishing *mean curvature.*

Minimal surfaces: Parametric surfaces Nonparametric surfaces are clearly too restrictive from the geometrical point of view and one is led to consider *parametric surfaces*. These are sets $\Sigma \subset \mathbb{R}^{n+1}$ so that there exist a connected open set $\Omega \subset \mathbb{R}^n$ and a map $v : \overline{\Omega} \to \mathbb{R}^{n+1}$ such that

$$\Sigma = v\left(\overline{\Omega}\right) = \left\{v\left(x\right) : x \in \overline{\Omega}\right\}.$$

For example, when $n = 2$ and $v = v\left(x_1, x_2\right) \in \mathbb{R}^3$, if we denote by $v_{x_1} \times v_{x_2}$ the normal to the surface (where $a \times b$ stands for the vectorial product of $a, b \in \mathbb{R}^3$ and $v_{x_1} = \partial v/\partial x_1$, $v_{x_2} = \partial v/\partial x_2$) we find that the area is given by

$$\text{Area}\left(\Sigma\right) = J\left(v\right) = \iint_{\Omega} \left|v_{x_1} \times v_{x_2}\right| dx_1 dx_2 .$$

Isoperimetric inequality: We begin with the simpler case of dimension two first.

Isoperimetric inequality in dimension two Let $A \subset \mathbb{R}^2$ be a bounded open set whose boundary, ∂A, is a sufficiently regular simple closed curve. Denote by $L\left(\partial A\right)$ the length of the boundary and by $M\left(A\right)$ the measure (the area) of A. The isoperimetric inequalitystates that

$$\left[L\left(\partial A\right)\right]^2 - 4\,\pi\, M\left(A\right) \geq 0.$$

Equality holds if and only if A is a disk (i.e. ∂A is a circle).
We can rewrite this into our formalism (here $n = 1$ and $N = 2$) by parametrizing the curve

$$\partial A = \left\{u\left(x\right) = \left(u^1\left(x\right), u^2\left(x\right)\right) : x \in \left[a, b\right]\right\}$$

and setting

$$L\left(\partial A\right) = L\left(u\right) = \int_a^b \sqrt{\left(\left(u^1\right)'\right)^2 + \left(\left(u^2\right)'\right)^2},$$

$$M\left(A\right) = M\left(u\right) = \frac{1}{2}\int_a^b \left(u^1\left(u^2\right)' - u^2\left(u^1\right)'\right) = \int_a^b u^1\left(u^2\right)' .$$

The problem is then to show that

$$(P) \quad \inf\left\{L\left(u\right) : M\left(u\right) = 1;\ u\left(a\right) = u\left(b\right)\right\} = 2\sqrt{\pi}.$$

Isoperimetric inequality in any dimension Isoperimetric inequality holds in any dimension. For open sets $A \subset \mathbb{R}^n$ with sufficiently regular boundary, ∂A, and it reads as

$$\left[L\left(\partial A\right)\right]^n - n^n\,\omega_n\left[M\left(A\right)\right]^{n-1} \geq 0$$

where ω_n is the measure of the unit ball of $\mathbb{R}^n$, $M\left(A\right)$ stands for the measure of A and $L\left(\partial A\right)$ for the $(n-1)$ measure of ∂A. Moreover, if A is sufficiently regular (for example, convex), there is equality if and only if A is a ball.