Introduction to the Calculus of Variations

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Spring Semester 2021

1 Course contents and outline

Our goal is to cover the following topics.

- **Introduction**
  - Finding minima of a function: Classical method
  - Finding minima of a function: Direct methods
  - Comparison of the two methods
  - Functions to functionals: The Calculus of Variations
  - History and classical problems: A brief historical tour of the origins of the calculus of variations. Examples of classical problems: The Brachistochrone, the Tautochrone, Fermat's principle of least time, least action principle, isoperimetric inequality, minimal surfaces of revolutions, minimal surfaces etc.
- **Classical Methods**
  - Fundamental lemma of Calculus of variations
  - Euler-Lagrange equations,
  - Lagrangian and Hamiltonian formulations,
  - Rudiments of convex analysis and Legendre transform,
  - Hamilton's equations
  - First Integrals: Symmetry and Noether's theorem
  - Hamilton-Jacobi equations,
  - Constrained problems and Lagrange multipliers

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  - Second variations and Jacobi fields
  - Example and counterexamples
  - An illustration of the classical methods: Geodesic curves.

- **Tools from analysis**
  - Recap of Preliminaries:
    * weak and weak* topologies, Reflexive Banach spaces,
    * convex sets and weak topology,
    * Banach-Alaoglu theorem,
    * Mazur's lemma and sequential weak lower semicontinuity of convex functions.
  - L^p and Sobolev spaces,
  - Some properties of Sobolev spaces
  - Poincaré and Sobolev inequalities.

- **Direct Methods**
  - **Dirichlet integral and p-Dirichlet Integral**
  - **Existence of minimizers:**
    * Existence theorem for convex functionals without lower order terms
    * Existence theorem for convex functional with lower order terms
    * Examples and counterexamples.
  - Derivation of the weak forms of the Euler-Lagrange equations, Dirichlet Principle
  - Discussion on necessity of convexity in the vectorial case, If time permits, derivation of quasiconvexity as a necessary condition for lower semicontinuity.
  - polyconvexity: weak continuity of determinants.

- **Regularity**
  - Weyl's lemma,
  - Statement of Schauder and L^p estimates
  - Proof of $W^{2,2}$ regularity (if time permits),
  - Brief mention of Hilbert's 19th Problem, De Giorgi's solution, De Giorgi-Nash-Moser estimates and the regularity for the p-Dirichlet integral (if time permits).

- **Area functional: Plateau's problem and minimal surfaces**
  - Parametric Plateau's problem: Douglas-Courant-Tonelli method
  - Reguality, uniqueness and nonuniqueness.
  - Nonparametric minimal surfaces.
  - Isoperimetric inequality

2 Main references

The basic references for this course are [3] and [5]. Advanced references which discuss most of our topics are [2] and [7]. An excellent references for minimal surfaces and CMC surfaces is [6]. For regularity questions, both [1] and [4] are excellent, albeit advanced references.

3 Prerequisites

- **Analysis** A course in Real analysis and a course in Measure and Integration is crucial. These would be used quite freely. Courant-Lebesgue lemma would be stated in the course, but not proved.
- **Functional analysis** A first course in functional analysis would be needed. Weak and weak* topologies, Reflexive Banach spaces, Hanhn-Banach theorem and its consequences for convex sets and weak topology, Banach-Alaoglu theorem, Mazur's lemma and sequential weak lower semicontinuity of convex functions would be used. However, most results concerning functional analysis preliminaries would be stated in the course, but not proved.
- **Function spaces** L^p spaces would be used constantly. A familiarity with Sobolev spaces would be useful, but not essential. We would define the Sobolev spaces and state their main properties and related inequalities in the course.
- **Geometry** Differntial geometry of curves and surfaces in $\mathbb{R}^n$, parametrization, parametrization and calculation of hypersurfaces which are graphs would be needed.
- **ODE** The Cauchy-Peano-Picard-Lindelöf-Lischitz theory of existence of solutions to Ordinary Differntial Equations would be used to prove the existence of geodesics. The existence theorem however would be stated in the course without proof.

References

[1] Ambrosio, L., Carlotto, A., and Massaccesi, A. *Lectures on elliptic partial differential equations*, vol. 18 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2018.

[2] Dacorogna, B. *Direct methods in the calculus of variations*, second ed., vol. 78 of *Applied Mathematical Sciences*. Springer, New York, 2008.

[3] Dacorogna, B. *Introduction to the calculus of variations*, third ed. Imperial College Press, London, 2015.

[4] Giaquinta, M., and Martinazzi, L. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, second ed., vol. 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2012.

[5] Jost, J., and Li-Jost, X. *Calculus of variations*, vol. 64 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.

[6] Struwe, M. *Plateau's problem and the calculus of variations*, vol. 35 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1988.

[7] Struwe, M. *Variational methods*, fourth ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.