Introduction to the Calculus of Variations
Problem Sheet 5

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1. Show that the Hamilton-Jacobi equations
$$S_t + H\left(t, u, S_u\right) = 0$$
can be solved by integration if it can be reduced to the form
$$\Phi\left(t, S_t\right) + \Psi\left(u, S_u\right) = 0,$$
for some functions Φ and Ψ.

2. Write down and solve the Hamilton-Jacobi equation corresponding to the functional
$$I\left[u\right] = \int_a^b \dot{u}^2\left(t\right)\ \mathrm{d}t,$$
and use the result to find the extremals of $I\left[u\right].$

3. Write down and solve the Hamilton-Jacobi equation corresponding to the functional
$$I\left[u\right] = \int_a^b \sqrt{1+\dot{u}^2\left(t\right)}\ \mathrm{d}t,$$
and use the result to find the extremals of $I\left[u\right].$

4. Write down and solve the Hamilton-Jacobi equation corresponding to the functional
$$I\left[u\right] = \int_a^b f\left(u\left(t\right)\right)\sqrt{1+\dot{u}^2\left(t\right)}\ \mathrm{d}t,$$
for some given function $f : \mathbb{R} \to \mathbb{R}$, and use the result to find the extremals of $I\left[u\right].$

5. Write down and solve the Hamilton-Jacobi equation corresponding to the functional
$$I\left[u\right] = \int_a^b \sqrt{t^2+u^2\left(t\right)}\sqrt{1+\dot{u}^2\left(t\right)}\ \mathrm{d}t,$$
and use the result to find the extremals of $I\left[u\right].$
Hint: Try the ansatz $S = \frac{1}{2}\left(At^2 + 2Btu + Cu^2\right).$

6. Let

$$I\left[y\right] = \int_a^b \sqrt{\phi_1\left(x\right) + \phi_2\left(y\left(x\right)\right)} \sqrt{1 + \left(y'\right)^2\left(x\right)}\, \mathrm{d}x,$$

for some given functions $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$. Show that formally the extremals of I can be recovered from the equations

$$\int \frac{\mathrm{d}x}{\sqrt{\phi_1\left(x\right) - \alpha}} - \int \frac{\mathrm{d}y}{\sqrt{\phi_2\left(y\right) + \alpha}} = \beta,$$

where α, β are constants.