Introduction to the Calculus of Variations
Problem Sheet 4

Swarnendu Sil

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1. **Poisson Brackets** Let $f, g, h \in C^2\left([a,b]\times\mathbb{R}^N\times\mathbb{R}^N\right)$, written $f = f(t,u,v)$ and likewise for g and h. Let the *Poisson Bracket* of f and g be defined as
$$\{f,g\} := \langle f_u, g_v\rangle - \langle f_v, g_u\rangle.$$
Let $\lambda, \mu \in \mathbb{R}$. Show the following properties of Poisson Brackets.

   (a) **anticommutativity**
$$\{f,g\} = -\{g,f\}.$$

   (b) **bilinearity**
$$\{\lambda f + \mu g, h\} = \lambda\{f,h\} + \mu\{g,h\}.$$
$$\{h, \lambda f + \mu g\} = \lambda\{h,f\} + \mu\{h,g\}.$$

   (c) **Leibnitz rule**
$$\{fg,h\} = \{f,h\}g + f\{g,h\}.$$

   (d) **Jacobi identity**
$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$

   (e) **distributive law for derivatives**
$$\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}.$$

2. **New first integrals from known ones**

   (a) Let $\Phi_1, \Phi_2 \in C^2\left(\mathbb{R}^N\times\mathbb{R}^N\right)$, $\Phi_i = \Phi_i(u,v)$ for $i = 1,2$, be two first integrals of the Hamilton's equations with Hamiltonian $H = H(t,u,v) \in C^2\left([a,b]\times\mathbb{R}^N\times\mathbb{R}^N\right)$. Show that $\{\Phi_1, \Phi_2\}$ is yet another first integral for the Hamiltonian H.

Note that problem (1) above implies that $C^2\left(\mathbb{R}^N \times \mathbb{R}^N\right)$ is an **algebra** with Poisson Bracket as the product. This one implies that given a C^2 Hamiltonian H, the set of its first integrals which does not depend explicitly on t is a **closed subalgebra** of $C^2\left(\mathbb{R}^N \times \mathbb{R}^N\right)$. Show these statements.

(b) Does the previous conclusion hold if Φ_1, Φ_2 are allowed to depend explicitly on t? (i.e. when $\Phi_1, \Phi_2 \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\Phi_i = \Phi_i(t,u,v)$ for $i = 1,2$, are the two first integrals)

(c) Is $\{\Phi_1, \{\Phi_1, \Phi_2\}\}$ another first integral for H? Is $\{\Phi_2, \{\Phi_1, \Phi_2\}\}$? Are they distinct first integrals? Discuss separately for the hypotheses of (a) and (b) above.

3. **cyclic coordinates and conjugate momenta** Let $f = f(t,u,\xi)$, $f \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$ be a Lagrangian density which satisfies the assumptions of the Hamiltonian regularity lemma. Thus in particular, its associated Hamiltonian $H = H(t,u,v)$ is $C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$. Suppose u_i is a cyclic coordinates for f for some $1 \leq i \leq N$.

   (a) Show that
   $$\frac{\partial H}{\partial u_i} = 0.$$

   (b) Show using the Euler-Lagrange equations that f_{ξ_i} is a first integral.

   (c) Show using Noether's theorem that f_{ξ_i} is a first integral.

   (d) Show using Hamilton's equations that v_i is a first integral.

   (e) Show using (b) or (c) above that v_i is a first integral.

   (f) Show that
   $$\frac{\partial f}{\partial t} = \frac{\partial H}{\partial t}.$$
   Using this, deduce that
   $$\frac{\partial H}{\partial t} = 0 \Rightarrow H \text{ is a first integral,}$$
   using each of the following separately.
   - Noether's theorem.
   - First integral theorem for time independent first integrals.
   - First integral theorem for time dependent first integrals.
   - The Euler-Lagrange equations.
   - The Hamilton's equations.

4. Consider the motion of a particle in $\mathbb{R}^3$ with position $x = (x_1, x_2, x_3)$. Suppose the Lagrangian density
$$f(t,x,\xi) = \frac{1}{2}m\xi^2 - U(x),$$

is such that the potential energy function U is invariant under rotations about the x_1-axis. Write this statement in a precise mathematical form and compute using Noether's theorem the corresponding conserved quantity.

5. Consider a particle of mass m moving in $\mathbb{R}$ under the influence of an inverse quadratic potential. More precisely, the Lagrangian density is

$$f(t,x,\xi) = \frac{1}{2}m\xi^2 - \frac{\alpha}{x^2}, \qquad \alpha > 0.$$

Write down the EL equations. Find the symmetries and use Noether's theorem to find first integrals. Using those first integrals, solve the system completely. (Hint: This system has a scaling symmetry, find it.)

6. Consider the motion of a particle in $\mathbb{R}^3$ with position $x = (x_1, x_2, x_3)$. Suppose the Lagrangian density is

$$f(t,x,\xi) = \frac{1}{2}m\xi^2 - q\phi + q\langle \xi, A\rangle,$$

where $m > 0$, $q \in \mathbb{R}\setminus\{0\}$, $\phi : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ and $A : [a,b]\times\mathbb{R}^3 \to \mathbb{R}^3$. Deduce the EL equations. Find out the momenta conjugate to the x_is, $1 \leq i \leq 3$. What are the conserved quantities? Can you guess what physical system does this problem describe?

7. Consider the motion of a particle in $\mathbb{R}^3$ with position $x = (x_1, x_2, x_3)$. Suppose the Lagrangian density is

$$f(t,x,\xi) = mc^2\sqrt{\left(1 - \frac{\xi^2}{c^2}\right)} - q\phi + q\langle \xi, A\rangle,$$

where $m, c > 0$, $q \in \mathbb{R}\setminus\{0\}$, $\phi : [a,b]\times\mathbb{R}^3 \to \mathbb{R}$ and $A : [a,b]\times\mathbb{R}^3 \to \mathbb{R}^3$. Deduce the EL equations. Find out the momenta conjugate to the x_is, $1 \leq i \leq 3$. What are the conserved quantities? Can you guess what physical system does this problem describe?

8. Consider the one dimensional motion of a particle in $\mathbb{R}$ with position x Suppose the Lagrangian density is

$$f(t,x,\xi) = mc^2\sqrt{\left(1 - \frac{\xi^2}{c^2}\right)} - \frac{1}{2}kx^2,$$

where $m, c, k > 0$ are constants. Deduce the EL equations. Find out the momenta conjugate to x. What are the conserved quantities? Can you guess what physical system does this problem describe?

9. Consider the motion of a particle in $\mathbb{R}^3$ whose position we would express in polar coordinates. Consider the Lagrangian density

$$f\left(t, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - \frac{\alpha}{r}, \qquad \alpha > 0.$$

Write down the EL equations and find out the symmetries and the conserved quantities and interpret them.

10. Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \leq i \leq M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t. The Lagrangian density is

$$f(t, u, \xi) = \frac{1}{2} \sum_{i=1}^{M} m_i \xi_i^2 - U\left(t, u(t)\right),$$

where the potential energy function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$ for the configuration $u(t)$ is given as

$$U\left(t, u(t)\right) = \sum_{\substack{i \neq j, \\ 1 \leq i,j \leq M.}} V\left(|u_i - u_j|\right),$$

for some nonnegative function $V : \mathbb{R} \to \mathbb{R}$. Write down the EL equations and find out the symmetries and the conserved quantities and interpret them. You might want to introduce 'center of mass' coordinates.