Introduction to the Calculus of Variations
Problem Sheet 3

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1. **Convex functions**

   (a) Let $f \in C^1(\mathbb{R}^n)$. Show that the following assertions are equivalent.

   (i) f is convex.

   (ii) For every $x, y \in \mathbb{R}^n$, the following inequality holds

   $$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$

   (iii) For every $x, y \in \mathbb{R}^n$, the following inequality is valid

   $$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0.$$

   Such an inequality is called a *monotonicity inequality*, i.e. the statement says that the gradient of f is *monotone*.

   (iv) If $f \in C^2(\mathbb{R}^n)$, then the above statements are also equivalent to the following:

   For every $x, v \in \mathbb{R}^n$, the following inequality holds

   $$\left\langle \nabla^2 f(x) v, v \right\rangle \geq 0.$$

   (b) **Subgradient** (This is a slightly advanced topic for the course, so solving this exercise is not strictly required to follow the course. But its fun!)

   In view of (ii) above, we can define a notion of a 'gradient' for a convex function even when they are not differentiable.

   Definition 1. *A vector $v \in \mathbb{R}^n$ is called a* subgradient *of a* ***convex*** *function $f : \mathbb{R}^n \to \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ if*

   $$f(x) \geq f(x_0) + \langle v, x - x_0 \rangle \quad \text{for every } x \in \mathbb{R}^n.$$

   The set of all subgradients of f at a point $x_0 \in \mathbb{R}^n$ is called the subdifferential *of f at x_0 and is denoted $\partial f(x_0)$.*

i. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $x_0 \in \mathbb{R}^n$. Show that $\partial f(x_0)$ is a nonempty, compact, convex set.

ii. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two convex functions and $x_0 \in \mathbb{R}^n$. Show that $f + g$ is convex and

$$\partial (f+g)(x_0) = \partial f(x_0) + \partial g(x_0),$$

where the sum of the sets on the right is the Minkowski sum, i.e.

$$A + B := \{x + y : x \in A, y \in B\}.$$

iii. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $x_0 \in \mathbb{R}^n$. Show that f is differentiable at x_0 if and only if $\partial f(x_0)$ is a singleton set. Describe $\partial f(x_0)$ in this case.

iv. Calculate $\partial f(x)$ for every $x \in \mathbb{R}^n$ when $f(x) = |x|$.

v. Calculate $\partial f(x)$ for every $x \in \mathbb{R}^n$ when

$$f(x) = \max_{1 \leq i \leq n} \{|x_i|\}.$$

vi. Show that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has a global minima at $x_0 \in \mathbb{R}^n$ if and only if $0 \in \partial f(x_0)$. Combined with iii. above, what does this condition reduce to if f is differentiable at x_0?

2. **Legendre Transform**

(a) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\})$.

Show the following.

i. If $f \not\equiv +\infty$, then $f^* > -\infty$.

ii. f^* is convex.

iii. f^{**} is convex and $f^{**} \leq f$.

iv. If f is bounded below and finite, then f^{**} is its convex envelope.

v. If f is convex, bounded below and finite then $f^{**} = f$.

vi. $f^{***} = f^*$.

vii. If $f \in C^1(\mathbb{R}^n)$, convex and finite, then

$$f(x) + f^*(\nabla f(x)) = \langle \nabla f(x); x \rangle, \quad \forall\, x \in \mathbb{R}^n.$$

viii. If $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and if

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty$$

then $f^* \in C^1(\mathbb{R}^n)$. Moreover, if $f \in C^1(\mathbb{R}^n)$ and

$$f(x) + f^*(x^*) = \langle x^*; x \rangle$$

then

$$x^* = \nabla f(x) \quad \text{and} \quad x = \nabla f^*(x^*).$$

(b) i. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Calculate f^*.

ii. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \in (0,1) \\ +\infty & \text{otherwise.} \end{cases}$$

Calculate f^* and f^{**}.

3. Let $\alpha, \beta \in \mathbb{R}^N$ be two given vectors and $f = f(t, u, \xi) \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$ be a given function. Set

$$X = \left\{u \in C^1\left([a,b]; \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta\right\}$$

and consider the problem

$$(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t))\ \mathrm{dt} \right\} = m.$$

Suppose $\bar{u} \in X \cap C^2\left([a,b]; \mathbb{R}^N\right)$ is a minimizer for (P).

(a) **Du Bois-Raymond's equation** Show that $\bar{u}$ satisfies,

$$\frac{d}{dt}\left[f(t, \bar{u}(t), \dot{\bar{u}}(t)) - \langle \dot{\bar{u}}(t), f_\xi(t, \bar{u}(t), \dot{\bar{u}}(t))\rangle\right] = f_t(t, \bar{u}(t), \dot{\bar{u}}(t))$$

for every $t \in (a,b)$.

(b) **Beltrami identity** If f does not depend explicitly on t, show that the function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$\Phi(u, \xi) := f(u, \xi) - \langle \xi, f_\xi(u, \xi)\rangle$$

is a first integral.

4. **Brachistochrone** Write down the associated Hamiltonian system and formally solve the brachistochrone problem and show that the solution is a cycloid. As a recap, the variational problem is:

$$m = \inf \left\{ I(u) := \int_a^b f(u(x), u'(x))\ \mathrm{d}x : u \in \mathcal{A} \right\}, \tag{1}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x, u, \xi) = f(u, \xi) = \sqrt{\left(\frac{1 + \xi^2}{2gu}\right)}.$$

The class of admissible paths is

$$\mathcal{A} := \left\{ \begin{array}{l} u \in C^1([0,b]) : u(0) = 0,\ u(b) = \beta \\ \qquad\qquad\qquad\qquad \text{and } u(x) > 0 \text{ for all } x \in (0,b] \end{array} \right\}.$$

(Hint: Use Beltrami identity together with the Euler-Lagrange equations.)

5. **Minimal surface of revolution** Write down the associated Hamiltonian system and formally solve the minimal surfaces of revolution problem and show that the solution is a catenoid formed by revolving a catenary. As a recap, a surface of revolution is a surface of the form

$$v(t,x) = (t, u(t)\cos x, u(t)\sin x)$$

with fixed end points $u(a) = \alpha, u(b) = \beta$ with $\alpha, \beta > 0$. The variational problem is:

$$m = \inf\left\{ I(u) := \int_a^b f(u(t), \dot{u}(t))\ \mathrm{d}t : u \in \mathcal{A} \right\}, \tag{2}$$

where $n = N = 1$ and the form of the Lagrangian density is

$$f(x,u,\xi) = f(u,\xi) = 2\pi u\sqrt{1+\xi^2}.$$

The class of admissible curves is

$$\mathcal{A} := \left\{ \begin{array}{l} u \in C^1([a,b]) : u(a) = \alpha,\ u(b) = \beta \\ \qquad\qquad\qquad\qquad \text{and } u(x) > 0 \text{ for all } x \in [a,b] \end{array} \right\}.$$

(Hint: Use Beltrami identity together with the Euler-Lagrange equations.)

6. **Mechanics of system of point masses** Let $m_i > 0$ be the mass and $u_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of the i-th particles for $1 \le i \le M$. Let $u(t) := (u_1(t), \ldots, u_M(t)) \in \mathbb{R}^{3M}$ be the configuration at time t. The *potential energy* function for the configuration $u(t)$ is a given function $U : \mathbb{R}_+ \times \mathbb{R}^{3M} \to \mathbb{R}$. The variational problem:

$$m = \inf\left\{ I(u) := \int_0^T f(t, u(t), \dot{u}(t))\ \mathrm{d}t : u(0) = u_0, u(T) = v_0 \right\}, \tag{3}$$

where $n = 1, N = 3M$, u_0, v_0 given and the form of the Lagrangian density is

$$f(x,u,\xi) = T(\xi) - U(t, u(t)).$$

Here T is the *kinetic energy* and is given by

$$T(\xi) := \frac{1}{2}\sum_{i=1}^{M} m_i \xi_i^2.$$

(a) Derive the Euler-Lagrange equations and the associated Hamiltonian system.

(b) Show that along the integral curves, the Hamiltonian can be written as the sum of the potential and kinetic energies, i.e.

$$T(\dot{u}) + U(t, u(t)).$$

(c) Is the Hamiltonian function a first integral?

7. Generalize the theorem about criterion of being a first integral to time-dependent functions. More precisely, find necessary and sufficient conditions under which a function $\Phi \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$, $\Phi = \Phi(t, u, v)$, is a first integral of of the Hamilton's equations with Hamiltonian $H = H(t, u, v) \in C^2\left([a,b] \times \mathbb{R}^N \times \mathbb{R}^N\right)$.