Introduction to the Calculus of Variations
Problem Sheet 2

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1. **Fundamental Lemma of the Calculus of Variations**

   (a) Supply an elementary proof for the following weaker version of the fundamental lemma of the calculus of variations.

   Lemma 1. *Let $u \in C^0((a,b))$ be such that*

   $$\int_a^b u(t)\,\psi(t)\ \mathrm{d}t = 0, \tag{1}$$

   holds for all for every $\psi \in C_c^\infty((a,b))$. Then $u \equiv 0$ in (a,b).

   (b) Prove the following general version of the fundamental lemma of the calculus of variations.

   Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be open and $u, v \in L^1_{loc}\left(\Omega; \mathbb{R}^N\right)$ be such that*

   $$\int_\Omega \langle u(x), \psi(x)\rangle\ \mathrm{d}x = 0, \tag{2}$$

   holds for all for every $\psi \in C_c^\infty\left(\Omega; \mathbb{R}^N\right)$ satisfying the condition

   $$\int_\Omega \langle v(x), \psi(x)\rangle\ \mathrm{d}x = 0. \tag{3}$$

   Then there exists a number $\lambda \in \mathbb{R}$ such that we have $u = \lambda v$ a.e. in Ω.

   Note that if $u, v \in L^2$, the geometric content of the lemma becomes clear, which is the following: If u is orthogonal to all C_c^∞ functions that are orthogonal to v, then u must be along v.

   (c) Using the fundamental lemma, supply an elementary proof for the following one dimensional version of the Du Bois-Raymond's lemma.

   Lemma 3. *Let $a, b \in \mathbb{R}$ and let $u \in L^1_{loc}((a,b))$ be such that*

   $$\int_a^b u(t)\,\dot{\psi}(t)\ \mathrm{d}t = 0, \qquad \text{for every } \psi \in C_c^\infty((a,b)). \tag{4}$$

   Then $u =$ constant a.e. in (a,b).

(d) Prove the Du Bois-Raymond's lemma.

Lemma 4 (Du Bois-Raymond's lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be open bounded and connected and $u \in L^1_{loc}\left(\Omega;\mathbb{R}^N\right)$ be such that*

$$\int_\Omega \langle u(x), D_i\psi(x)\rangle \, \mathrm{d}x = 0, \qquad \text{for } 1 \le i \le n, \tag{5}$$

for every $\psi \in C_c^\infty\left(\Omega;\mathbb{R}^N\right)$. Then $u =$ constant a.e. in Ω.

2. **Euler-Lagrange equations**

(a) Generalize the Euler-Lagrange equation theorem for the following variational problem.

$$\inf_{u\in X}\left\{ I(u) = \int_a^b f\left(t, u(t), \dot{u}(t), \ldots, u^{(r)}(t)\right) \, \mathrm{d}t \right\},$$

where $r \ge 2$ is a positive integer, the notation $u^{(r)}$ means the r-th derivative of u and

$$X = \left\{ u \in C^r([a,b]) : u^{(j)}(a) = \alpha_j, u^{(j)}(b) = \beta_j, 1 \le j \le r-1 \right\}.$$

(b) Generalize the Euler-Lagrange equation theorem for the following variational problem.

$$\inf_{u\in X}\left\{ I(u) = \int_a^b f(t, u(t), \dot{u}(t)) \, \mathrm{d}t \right\},$$

where

$$X = \left\{ u \in C^1([a,b]) : u(a) = \alpha \right\}.$$

3. **Poincaré-Wirtinger inequality** Let $\lambda \in \mathbb{R}$ and $T > 0$. Set

$$X_{\mathrm{Dir},0} := \left\{ u \in C^1\left([0,T];\mathbb{R}^N\right) : u(0) = 0 = u(T) \right\}.$$

Consider the following variational problem

$$\inf\left\{ I(u) = \int_0^T f(t, u(t), \dot{u}(t)) \, \mathrm{d}t : u \in X_{\mathrm{Dir},0} \right\} = m,$$

where the Lagrangian density is

$$f(t,u,\xi) = \frac{1}{2}\xi^2 - \frac{\lambda^2}{2}u^2.$$

(a) Derive the Euler-Lagrange equations for this problem.

(b) Find all solutions of those equations.

(c) Discuss the value of m for different values of λ.

(d) Show the following inequality, which is a one dimensional version of what is called the Poincaré-Wirtinger inequality:

$$\int_0^T (u(t))^2 \, \mathrm{d}t \le \left(\frac{T}{\pi}\right)^2 \int_0^T (\dot{u}(t))^2 \, \mathrm{d}t.$$